Pismeni deo ispita iz KVANTNE STATISTIČKE FIZIKE

1. Posmatrajmo jednu slobodnu česticu mase m koja se nalazi u oblasti prostora zapremine V i koja je u ravnoteži sa termostatom temperature T.

   (a) Naći matrične elemente ravnotežnog statističkog operatora $\hat{\rho}$ koji opisuje stanje te čestice u impulsnoj reprezentaciji, $\rho(\mathbf{p}, \mathbf{p}') = \langle \mathbf{p}|\hat{\rho}|\mathbf{p}'\rangle$.

   (b) Naći matrične elemente ravnotežnog statističkog operatora $\hat{\rho}$ koji opisuje stanje te čestice u koordinatnoj reprezentaciji, $\rho(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r}|\hat{\rho}|\mathbf{r}'\rangle$.

   (c) Izračunati srednju energiju čestice na temperaturi T.

2. Posmatrajmo idealni gas fotona u zapremini V (trodimenzionalni prostor) i na temperaturi T. Disperziona relacija fotona je $\varepsilon_{\mathbf{k}} = \hbar c|\mathbf{k}|$, gde je $\mathbf{k}$ talasni vektor fotona.

   (a) Napisati izraz za (infinitezimalni) broj fotona $\mathrm{d}N_\lambda$ čije talasne duzine leže u infinitezimalnom intervalu talasnih dužina $(\lambda, \lambda + \mathrm{d}\lambda)$.

   (b) Izračunati energiju E, pritisak p i toplotni kapacitet C_V fotonskog gasa.

3. Cooper je 1956. godine pokazao da dva elektrona koji međusobno interaguju privlačnom interakcijom u prisustvu potpuno popunjenog Fermijevog mora na nuli temperature formiraju vezano stanje (tzv. Cooperov par). Izračunati energiju veze E_b Cooperovog para u osnovnom stanju, čiji je impuls centra mase jednak nuli.

   Zanemariti interakciju elektrona koji formiraju par sa elektronima iz Fermijevog mora. Uzeti da se orbitalni deo talasne funkcije osnovnog stanja para se može razviti po bazisu ravnih talasa kao (k_F je Fermijev talasni vektor)

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_{\substack{\mathbf{k} \\ |\mathbf{k}|>k_F}} g_{\mathbf{k}} \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_1} \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_2}.$$

   Matrični elementi potencijala međusobne interakcije posmatranog para elektrona (Ω je zapremina na koju se vrši normiranje)

$$V_{\mathbf{k}\mathbf{k}'} = \frac{1}{\Omega} \int \mathrm{d}^3\mathbf{r} \, \mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')\mathbf{r}} \, V(\mathbf{r})$$

   su oblika

$$V_{\mathbf{k}\mathbf{k}'} = \begin{cases} -V, & \varepsilon_F \leq \varepsilon_{\mathbf{k}}, \varepsilon_{\mathbf{k}'} \leq \varepsilon_F + \hbar\omega_c \\ 0, & \text{inače,} \end{cases}$$

   gde je $V > 0$, ε_F je Fermijeva energija, $\varepsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/(2m)$, dok je $\hbar\omega_c \ll \varepsilon_F$. Jednočestičnu gustinu stanja $N(\varepsilon_F)$ na Fermijevoj površi za jednu vrednost spina smatrati poznatom. Specijalno, napisati izraz za E_b u limesu $N(\varepsilon_F)V \ll 1$.

Zadatke pripremio
dr Veljko Janković

www.ipb.ac.rs

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1. $\hat{\mathcal{H}} = \frac{\hat{\vec p}^2}{2m}$, $\hat\varrho = \frac{1}{Z} e^{-\beta\hat{\mathcal{H}}}$, $Z = \mathrm{Tr}\{e^{-\beta\hat{\mathcal{H}}}\}$

(a) $\langle \vec p|\hat\varrho|\vec p\,'\rangle = \varrho(\vec p,\vec p\,') = \delta_{\vec p,\vec p\,'}\frac{1}{Z}e^{-\beta\frac{\vec p^2}{2m}}$

$$Z = \sum_{\vec p}\langle \vec p|e^{-\beta\hat{\mathcal{H}}}|\vec p\rangle = \frac{V}{(2\pi\hbar)^3}\int d^3\vec p\; e^{-\beta\frac{\vec p^2}{2m}} = \frac{V}{(2\pi\hbar)^3}\times\left[\int_{-\infty}^{+\infty} dp_x\, e^{-\beta\frac{p_x^2}{2m}}\right]^3$$

$$Z = \frac{V}{(2\pi\hbar)^3}\left(\sqrt{\frac{\pi}{\frac{\beta}{2m}}}\right)^3 = \frac{V}{(2\pi\hbar)^3}(2\pi m k_B T)^{3/2} = \boxed{V\left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3 = \frac{V}{\lambda_T^3} = Z}$$

$$\boxed{\langle \vec p|\hat\varrho|\vec p\,'\rangle = \delta_{\vec p,\vec p\,'}\frac{1}{V}\left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3 e^{-\beta\frac{\vec p^2}{2m}}}$$

(b) $$\varrho(\vec r,\vec r\,') = \langle\vec r|\hat\varrho|\vec r\,'\rangle = \sum_{\vec p,\vec p\,'}\langle\vec r|\vec p\rangle\langle\vec p|\hat\varrho|\vec p\,'\rangle\langle\vec p\,'|\vec r\,'\rangle = \sum_{\vec p}\langle\vec r|\vec p\rangle\langle\vec p|\hat\varrho|\vec p\rangle\langle\vec p|\vec r\,'\rangle$$

$$\varrho(\vec r,\vec r\,') = \sum_{\vec p}\frac{1}{\sqrt V}e^{i\frac{\vec p\vec r}{\hbar}}\frac{\lambda_T^3}{V}e^{-\beta\frac{\vec p^2}{2m}}\frac{1}{\sqrt V}e^{-\frac{i}{\hbar}\vec p\cdot\vec r\,'} = \frac{1}{V}\sum_{\vec p}\frac{\lambda_T^3}{V}e^{-\beta\frac{\vec p^2}{2m}+\frac{i}{\hbar}\vec p(\vec r-\vec r\,')}$$

$$= \int\frac{d^3\vec p}{(2\pi\hbar)^3}\frac{\lambda_T^3}{V}e^{-\beta\frac{\vec p^2}{2m}+\frac{i}{\hbar}\vec p(\vec r-\vec r\,')} = \frac{1}{(2\pi\hbar)^3}\frac{\lambda_T^3}{V}\left[\int_{-\infty}^{+\infty}dp_x\,e^{-\beta\frac{p_x^2}{2m}+\frac{i}{\hbar}p_x(x-x')}\right]\times\begin{bmatrix}p_x\to p_y\\ x\to y\\ x'\to y'\end{bmatrix}\times\begin{bmatrix}p_x\to p_z\\ x\to z\\ x'\to z'\end{bmatrix}$$

$$\varrho(\vec r,\vec r\,') = \frac{1}{(2\pi\hbar)^3}\frac{\lambda_T^3}{V}\sqrt{\frac{\pi}{\beta/(2m)}}^{\,3}\exp\left[-\frac{(\vec r-\vec r\,')^2}{\hbar^2\cdot 4\frac{\beta}{2m}}\right]$$

$$\boxed{\varrho(\vec r,\vec r\,') = \frac{1}{V}\exp\left[-\frac{mk_BT}{2\hbar^2}(\vec r-\vec r\,')^2\right]}$$

(c) $$\langle\hat{\mathcal{H}}\rangle = \mathrm{Tr}(\hat{\mathcal{H}}\hat\varrho) = \frac{1}{Z}\mathrm{Tr}(\hat{\mathcal{H}}e^{-\beta\hat{\mathcal{H}}}) = \frac{1}{Z}\left(-\frac{\partial}{\partial\beta}\right)\mathrm{Tr}\left(e^{-\beta\hat{\mathcal{H}}}\right) = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\partial(\ln Z)}{\partial\beta}$$

$$Z = \frac{V}{h^3}\left(\frac{2\pi m}{\beta}\right)^{3/2} \Rightarrow \frac{\partial Z}{\partial\beta} = \frac{V}{h^3}(2\pi m)^{3/2}\left(-\frac{3}{2}\right)\beta^{-5/2} = -\frac{3}{2\beta}Z$$

$$\Rightarrow \boxed{\langle\hat{\mathcal{H}}\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{3}{2\beta} = \frac{3}{2}k_BT}$$

2. (a) $$N = \sum_{\vec k\sigma}\bar n_{\vec k\sigma} = \int d\varepsilon\, g(\varepsilon)\,\bar n(\varepsilon)$$

dva stanja polarizacije

$$g(\varepsilon) = \sum_{\vec k\sigma}\delta(\varepsilon-\varepsilon_{\vec k\sigma}) = \sum_{\vec k\sigma}\delta(\varepsilon-\hbar ck) = \frac{2V}{(2\pi)^3}\times 4\pi\int_0^{+\infty}dk\,k^2\,\delta(\varepsilon-\hbar ck)$$

$$\boxed{g(\varepsilon) = \frac{V}{\pi^2}\frac{1}{(\hbar c)^3}\varepsilon^2\,\Theta(\varepsilon)}$$

$$N = \int_0^{+\infty}d\varepsilon\,\frac{V}{\pi^2(\hbar c)^3}\varepsilon^2\frac{1}{e^{\beta\varepsilon}-1}\quad,\quad \varepsilon = \hbar\omega = \frac{h}{2\pi}2\pi\frac{c}{\lambda} = \frac{hc}{\lambda}$$

$$d\varepsilon = -\frac{hc}{\lambda^2}d\lambda$$

$$N = \int_0^{+\infty}\frac{hc}{\lambda^2}d\lambda\,\frac{V}{\pi^2(\hbar c)^3}\left(\frac{hc}{\lambda}\right)^2\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}$$

$$\frac{hc}{\lambda^2}\frac{V}{\pi^2\left(\frac{h}{2\pi}c\right)^3}\left(\frac{hc}{\lambda}\right)^2 = \frac{1}{\lambda^4}V\frac{(2\pi)^3}{\pi^2} = \frac{8\pi V}{\lambda^4}$$

$$N = \int_0^{+\infty}d\lambda\,\frac{8\pi V}{\lambda^4}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1} \Rightarrow \boxed{dN_{(\lambda,\lambda+d\lambda)} = d\lambda\,\frac{8\pi V}{\lambda^4}\left[\exp\left(\frac{hc}{\lambda k_BT}\right)-1\right]^{-1}}$$

(b) $$E = \int d\varepsilon\,\varepsilon\,g(\varepsilon)\,\bar n(\varepsilon) = \int_0^{+\infty}d\varepsilon\,\varepsilon\frac{V}{\pi^2}\frac{1}{(\hbar c)^3}\varepsilon^2\frac{1}{e^{\beta\varepsilon}-1} = \{\beta\varepsilon = x\}$$

$$= \frac{V}{\pi^2}\frac{1}{(\hbar c)^3}(k_BT)^4\int_0^{+\infty}dx\,\frac{x^3}{e^x-1} = V\left(\frac{k_BT}{\hbar c}\right)^3\cdot k_BT\cdot\underbrace{\Gamma(4)}_{3!=6}\underbrace{\zeta(4)}_{\frac{\pi^4}{90}}\frac{1}{\pi^2} = \boxed{\frac{\pi^2}{15}V\left(\frac{k_BT}{\hbar c}\right)^3k_BT = E}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \quad , \quad C_V = V \frac{\pi^2}{15} \frac{1}{(\hbar c)^3} k_B^4 \cdot 4T^3 = \boxed{V\left(\frac{k_B T}{\hbar c}\right)^3 \cdot \frac{4\pi^2}{15} k_B = C_V}$$

$$pV = \frac{1}{3}E \quad , \quad \boxed{p = \frac{E}{3V} = \frac{\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3 k_B T}$$

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3. Schrödingerova jednadžba za 2 elektrona

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}_1^2 - \frac{\hbar^2}{2m}\vec{\nabla}_2^2 + V(\vec{r}_1 - \vec{r}_2)\right)\psi_0(\vec{r}_1, \vec{r}_2) = E_0 \psi_0(\vec{r}_1, \vec{r}_2)$$

$$\psi_0(\vec{r}_1, \vec{r}_2) = \sum_{\vec{k}}{}' g_{\vec{k}}\, e^{i\vec{k}\vec{r}_1} e^{-i\vec{k}\vec{r}_2} \quad , \quad \sum_{\vec{k}}{}' = \sum_{\substack{\vec{k} \\ |\vec{k}| > k_F}}$$

$$\sum_{\vec{k}}{}'\left(2\varepsilon_{\vec{k}}\, g_{\vec{k}}\, e^{i\vec{k}\vec{r}_1} e^{-i\vec{k}\vec{r}_2} + V(\vec{r}_1 - \vec{r}_2)\, g_{\vec{k}}\, e^{i\vec{k}\vec{r}_1} e^{-i\vec{k}\vec{r}_2}\right) = \sum_{\vec{k}}{}' E_0\, g_{\vec{k}}\, e^{i\vec{k}\vec{r}_1} e^{-i\vec{k}\vec{r}_2}$$

$$\sum_{\vec{k}}{}' g_{\vec{k}}\, V(\vec{r}_1 - \vec{r}_2)\, e^{i\vec{k}\vec{r}_1} e^{-i\vec{k}\vec{r}_2} = \sum_{\vec{k}}{}' g_{\vec{k}}\,(E_0 - 2\varepsilon_{\vec{k}})\, e^{i\vec{k}\vec{r}_1} e^{-i\vec{k}\vec{r}_2}$$

$\underline{\vec{r}_1 - \vec{r}_2 = \vec{\rho}}$:

$$\sum_{\vec{k}}{}' g_{\vec{k}}\, V(\vec{\rho})\, e^{i\vec{k}\vec{\rho}} = \sum_{\vec{k}}{}' g_{\vec{k}}\,(E_0 - 2\varepsilon_{\vec{k}})\, e^{i\vec{k}\vec{\rho}}$$

$$\int d\vec{\rho}\, e^{-i\vec{k}'\vec{\rho}} \sum_{\vec{k}}{}' g_{\vec{k}}\, V(\vec{\rho})\, e^{i\vec{k}\vec{\rho}} = \sum_{\vec{k}}{}' g_{\vec{k}}\,(E_0 - 2\varepsilon_{\vec{k}}) \underbrace{\boxed{\int d\vec{\rho}\, e^{-i\vec{k}'\vec{\rho}} e^{i\vec{k}\vec{\rho}}}}_{\Omega\, \delta_{\vec{k}'\vec{k}}}$$

$$\sum_{\vec{k}}{}' g_{\vec{k}} \underbrace{\int d\vec{\rho}\, e^{-i\vec{k}'\vec{\rho}}\, V(\vec{\rho})\, e^{i\vec{k}\vec{\rho}}}_{\Omega V_{\vec{k}'\vec{k}}} = \Omega\, g_{\vec{k}'}\,(E_0 - 2\varepsilon_{\vec{k}'})$$

$$\boxed{g_{\vec{k}} = \frac{1}{E_0 - 2\varepsilon_{\vec{k}}} \sum_{\vec{k}'}{}' V_{\vec{k}\vec{k}'}\, g_{\vec{k}'}}$$

$$V_{\vec{k}\vec{k}'} = \begin{cases} -V, & \varepsilon_F \le \varepsilon_{\vec{k}}, \varepsilon_{\vec{k}'} \le \varepsilon_F + \hbar\omega_c \\ 0, & \text{inače} \end{cases}$$

$\underline{\varepsilon_F \le \varepsilon_{\vec{k}} \le \varepsilon_F + \hbar\omega_c}$ $\Rightarrow g_{\vec{k}} = \frac{1}{E_0 - 2\varepsilon_{\vec{k}}} \sum_{\vec{k}'}{}'' (-V)\, g_{\vec{k}'}$, $\sum_{\vec{k}'}{}'' = \sum_{\vec{k}'}\Big|_{\varepsilon_F \le \varepsilon_{\vec{k}'} \le \varepsilon_F + \hbar\omega_c}$

$\underline{\varepsilon_{\vec{k}} > \hbar\omega_c + \varepsilon_F}$ $\Rightarrow g_{\vec{k}} = 0$

$$g_{\vec{k}} = \frac{V}{2\varepsilon_{\vec{k}} - E_0} \sum_{\vec{k}'}{}'' g_{\vec{k}'}$$

$$\sum_{\vec{k}}{}'' g_{\vec{k}} = \left[\sum_{\vec{k}}{}'' \frac{V}{2\varepsilon_{\vec{k}} - E_0}\right]\left[\sum_{\vec{k}'}{}'' g_{\vec{k}'}\right]$$

$$\Rightarrow \frac{1}{V} = \sum_{\vec{k}}{}'' \frac{1}{2\varepsilon_{\vec{k}} - E_0}$$

$$\frac{1}{V} = \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_c} d\varepsilon\, N(\varepsilon)\, \frac{1}{2\varepsilon - E_0} \approx N(\varepsilon_F) \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_c} \frac{d\varepsilon}{2\varepsilon - E_0}$$

$$\frac{1}{N(\varepsilon_F) V} = \frac{1}{2} \ln\left(\frac{2(\varepsilon_F + \hbar\omega_c) - E_0}{2\varepsilon_F - E_0}\right)$$

$$\frac{2(\varepsilon_F + \hbar\omega_c) - E_0}{2\varepsilon_F - E_0} = e^{\frac{2}{N(\varepsilon_F)V}}$$

$$2(\varepsilon_F + \hbar\omega_c) - E_0 = 2\varepsilon_F\, e^{\frac{2}{N(\varepsilon_F)V}} - E_0\, e^{\frac{2}{N(\varepsilon_F)V}}$$

$$E_0\left(e^{\frac{2}{N(\varepsilon_F)V}} - 1\right) = 2\varepsilon_F\left(e^{\frac{2}{N(\varepsilon_F)V}} - 1\right) - 2\hbar\omega_c$$

$$E_0 = 2\varepsilon_F - 2\hbar\omega_c\left(e^{\frac{2}{N(\varepsilon_F)V}} - 1\right)^{-1}$$

$$\Rightarrow \boxed{E_b = 2\hbar\omega_c\left(\exp\left(\frac{2}{N(\varepsilon_F)V}\right) - 1\right)^{-1}}$$

u limesu $N(\varepsilon_F) V \ll 1$

$$\boxed{E_b \approx 2\hbar\omega_c\, e^{-\frac{2}{N(\varepsilon_F)V}}}$$