Variational Methods with and without Replicas for a Zero-Dimensional Disorder Model

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- Hamilton Function

$$H(x) = \frac{1}{2}\kappa x^2 + \varepsilon U(x)$$

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- Disorder Average

$$\overline{\bullet} = \int \mathcal{D}U \bullet \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, R^{-1}(x - x')U(x)U(x')\right]$$

Thermodynamic Properties

- Partition Function

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- Problem

$$\log \overline{Z} \neq \overline{\log Z}$$

Generating Random Potentials

- Randomization Method [1]

$$U(x) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} [A_n cos(k_n x) + B_n sin(k_n x)]$$

- J. Majda et al., Phys. Rep. **314** (1999) 237

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- Wave Numbers

$$p(k_n) = \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ik_n x} R(x)}{R(0)}$$

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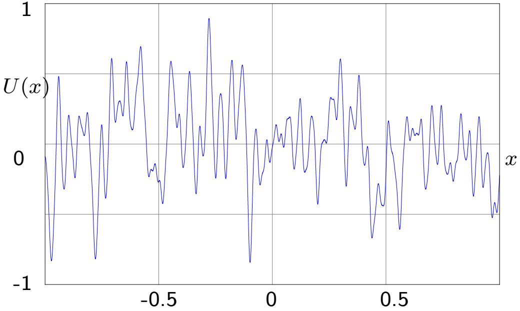
Examples

- Example for Random Potential with Gauss Correlation



Gaussian correlation with parameters

$N = 100$ and $\lambda = 1$.

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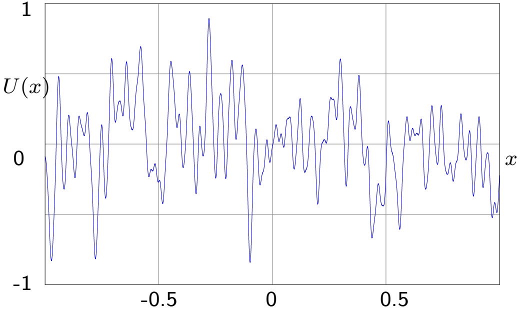
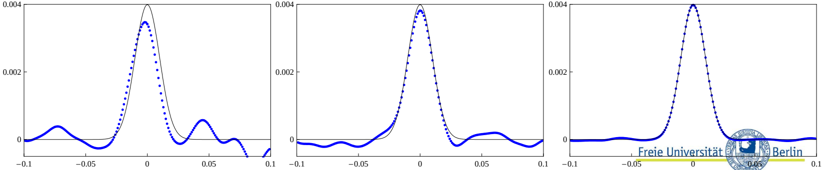


Gaussian correlation with parameters $N = 100$ and $\lambda = 1$.

- Comparison of numerical averages of $U(x)U(0)$ with $R(x)$.



Choice of parameter $N = 100$. The product $U(0)U(x)$ is averaged over 100, 1 000, and 10 000, respectively.

Replica Trick

- Replicated Partition Function

$$\overline{Z^n} = \overline{\left\{ \int_{-\infty}^{\infty} dx \exp[-\beta H(x)] \right\}^n} = \int d^n x \exp\left[-\beta H_r(\mathbf{x})\right]$$

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- Non-Random Hamilton Function

$$H_r(\mathbf{x}) = \frac{1}{2}\kappa \sum_{a=1}^{n} x_a^2 - \frac{1}{2}\beta\varepsilon^2 \sum_{a,b} R(x_a - x_b)$$

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- Trick: $\overline{\log Z} = \overline{\frac{1}{n}\log Z^n} = \lim_{n\to 0} \frac{\overline{Z^n}-1}{n} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n}$

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- Variational Perturbation Expansion:

$$H_r(x) = H_t(\mathbf{x}) + [H_r(\mathbf{x}) - H_t(\mathbf{x})]$$

$$H_t(\mathbf{x}) = \frac{1}{2} \sum_{a,b} x_a G_{ab}^{-1} x_b \;, \quad G_{ab}^{-1} = \kappa\delta_{ab} - \sigma_{ab}$$

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Perturbation Expansion

- Cumulant Expansion

$$F_n^N = F_t - \frac{1}{\beta} \sum_{k=1}^{N} \frac{(-\beta)^k}{k!} \left\langle [H_r(\mathbf{x}) - H_t(\mathbf{x})]^k \right\rangle_{H_t}^{c}$$

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- First-Order Free Energy

$$F_n^{(1)} = -\frac{n}{2\beta}\log\left(\frac{2\pi e}{\beta}\right) + \frac{1}{2\beta}\mathrm{Tr}\log G^{-1} + \frac{\kappa}{2\beta}\mathrm{Tr}G - \frac{\beta}{2}\sum_{a,b} f\left[q(a,b)\right],$$

$$f[q(a,b)] = f\left[\frac{1}{\beta}(G_{aa} + G_{bb} - 2G_{ab})\right] = \langle R(x_a - x_b)\rangle_{H_r}$$

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- Conditional Equations due to Saddle-Point Equation

$$\frac{\partial F_n^{(1)}}{\partial \sigma_{ab}} = 0 \Rightarrow \quad \begin{array}{ccc} c \neq d & \Rightarrow & \sigma_{cd} = -2\beta f_\xi[q(c,d)] \\ c = d & \Rightarrow & \sigma_{cc} = -\sum_{a\neq c}\sigma_{ac} \end{array}$$

Parisi Matrices

- $n = 9,\ k = 0$: Replica Symmetry

$$\begin{pmatrix} \tilde{p} & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 \\ p_0 & \tilde{p} & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 \\ p_0 & p_0 & \tilde{p} & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 \\ p_0 & p_0 & p_0 & \tilde{p} & p_0 & p_0 & p_0 & p_0 & p_0 \\ p_0 & p_0 & p_0 & p_0 & \tilde{p} & p_0 & p_0 & p_0 & p_0 \\ p_0 & p_0 & p_0 & p_0 & p_0 & \tilde{p} & p_0 & p_0 & p_0 \\ p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & \tilde{p} & p_0 & p_0 \\ p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & \tilde{p} & p_0 \\ p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & \tilde{p} \end{pmatrix}$$

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- $n = 9,\ k = 1,\ m_1 = 3$: 1-step Replica-Symmetry Breaking

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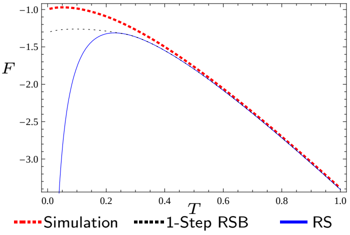
- Reparametrization as Continuous Function

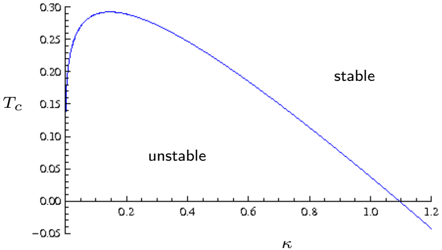
$$p(u) = p_i \ , \quad \text{if } m_{i+1} < u < m_i$$

Results

- Replica-Symmetry + Replica-Symmetry Breaking $(k = 0, 1)$



- Almeida-Thouless-Line: Replica Symmetry



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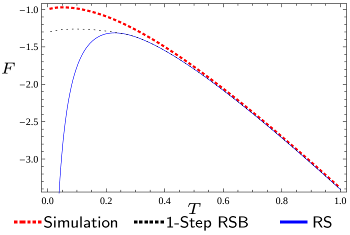

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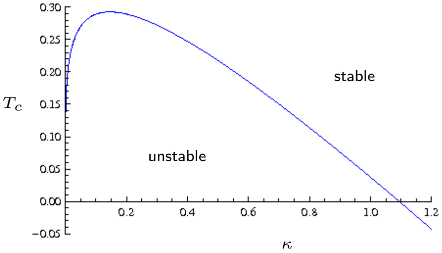

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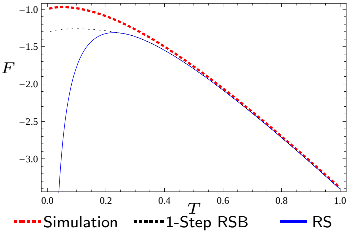

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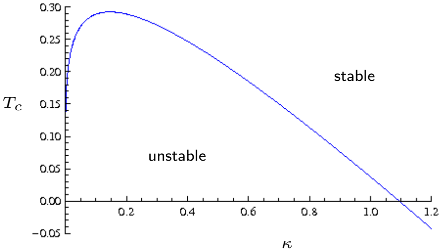

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[2] A. Engel, Nucl. Phys. B **410** (1993) 617

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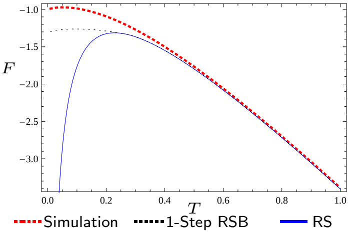

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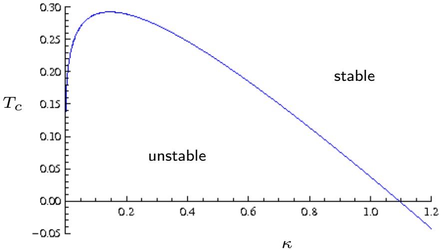

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- Numerical Calculations with higher steps k show no improvement

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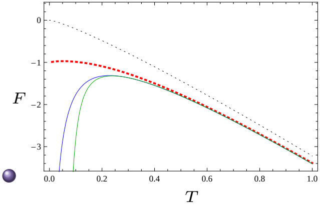
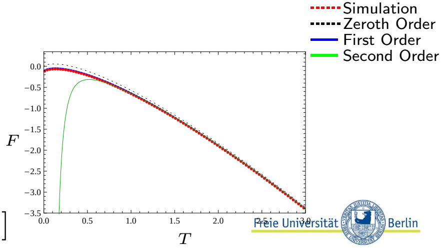
- 



Temperature Variation

- Artificially introducing variational parameter:
$T \to \tau\sqrt{1+\varepsilon r}\ , \quad r = \frac{\tau^2 - T^2}{\varepsilon\tau^2}\ , \quad \frac{\partial}{\partial\tau} F_N(T,\tau) = 0$

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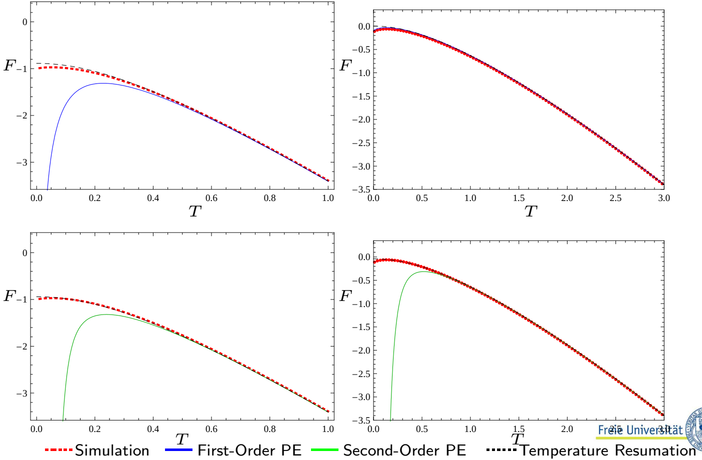
- Taylor-Expansion to corresponding order yields



The parameters are $\varepsilon = 1$ and $\kappa = 0.01$, $\kappa = 0.5$, $\kappa = 2$.