Nonlinear BEC Dynamics by Harmonic Modulation of s-wave Scattering Length*

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Overview

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  - BEC with modulated interaction - recent experiment
- Theoretical background
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- Spherically symmetric BEC
  - Condensate dynamics
  - Excitation spectra
  - Perturbative approach
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- Cylindrically symmetric BEC
  - Condensate dynamics
  - Excitation spectra
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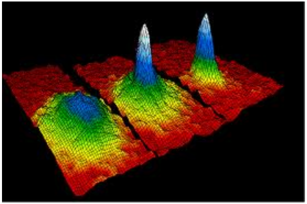
Experiments with ultracold atoms

- Intensive progress in the field of ultracold atoms has been recognized by Nobel prize for physics in 2001
- Cold alkali atoms:
  Rb, Na, Li, K ...
  $T \sim 1\, nK$, $\rho \sim 10^{14}\, cm^{-3}$



- Cold bosons, cold fermions
- Harmonic trap, optical lattice
- Short-range interactions,
  long-range dipolar interactions
- Tunable quantum systems concerning dimensionality, type and strength of interactions

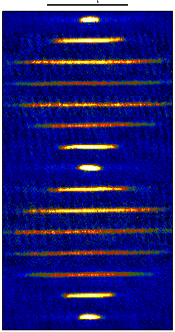
BEC with modulated interaction

- Motivation - recent experiment by Randy Hulet's group at Rice University and by Vanderlai Bagnato's group at São Paulo University: PRA **81**, 053627 (2010)

- BEC of ^{7}Li is confined in a cylindrical trap

- Time-dependent modulation of atomic interactions via a Feshbach resonance

- Excitation of the lowest-lying quadrupole mode



- Interesting setup for studying nonlinear BEC dynamics

Mean-field description

- Gross-Pitaevskii equation assuming $T = 0$ (no thermal excitations)

$$\imath\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\Delta + V(\vec{r}) + g|\psi(\vec{r},t))|^2\right]\psi(\vec{r},t)$$

- $\psi(\vec{r},t)$ is a condensate wave-function
- $V(\vec{r}) = \frac{1}{2}m\omega_\rho^2(\rho^2 + \lambda^2 z^2)$ is a harmonic trap potential, $l = \sqrt{\hbar/m\omega_\rho}$ is a characteristic harmonic oscillator length
- effective interaction between atoms is given by $g\delta(\vec{r})$
- $g = \frac{4\pi\hbar^2 Na}{m}$, a is $\boldsymbol{s}$-wave scattering length, N is number of atoms in the condensate

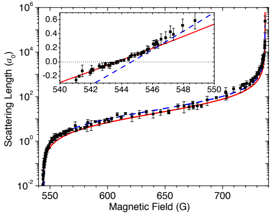
Feshbach resonance

- Scattering length depends on external magnetic field

- PRL **102**, 090402: ^{7}Li

$$a(B) = a_{\mathrm{BG}}\left(1 + \frac{\Delta}{B - B_\infty}\right)$$

$a_{\mathrm{BG}} = -24.5\, a_0$, $B_\infty = 736.8\,\mathrm{G}$,
$\Delta = 192\, G$



- Scattering length can be modulated using external magnetic field via a Feshbach resonance

$$B(t) = B_{\mathrm{av}} + \delta B \cos \Omega t, \quad a(t) \simeq a_{\mathrm{av}} + \delta a \cos \Omega t$$

$$a_{\mathrm{av}} = a(B_{\mathrm{av}}), \quad \delta a = -\frac{a_{\mathrm{BG}} \Delta \delta B}{(B_{\mathrm{av}} - B_\infty)^2}$$

$B_{\mathrm{av}} = 565\,\mathrm{G}$, $\delta B = 14\,\mathrm{G}$, $a_{\mathrm{av}} \sim 3a_0$, $\delta a \sim 2a_0$

Gaussian approximation (1)

- To simplify calculations and to obtain analytical insight, we approximate density of atoms by a Gaussian
- For an axially symmetric trap

$$\psi(\rho, z, t) = C(t) \exp\left[-\frac{1}{2}\frac{\rho^2}{u(t)^2} + \imath\rho^2 A_u(t)\right] \exp\left[-\frac{1}{2}\frac{z^2}{v(t)^2} + \imath z^2 A_v(t)\right]$$

- By extremizing corresponding action, we obtain two ordinary differential equations, PRL **77**, 5320 (1996)
- In the dimensionless form

$$\frac{d^2u(t)}{dt^2} + u(t) - \frac{1}{u(t)^3} - \frac{p(t)}{u(t)^3 v(t)} = 0$$
$$\frac{d^2v(t)}{dt^2} + \lambda^2 v(t) - \frac{1}{v(t)^3} - \frac{p(t)}{u(t)^2 v(t)^2} = 0$$

- Interaction: $p(t) = \sqrt{\frac{2}{\pi}} N a(t)/l$

Gaussian approximation (2)

- Using this type of approximation and relying on the linear stability analysis, frequencies of low-lying collective modes have been analytically calculated
- Equilibrium widths

$$u_0 = \frac{1}{u_0^3} + \frac{p}{u_0^3 v_0}, \quad \lambda^2 v_0 = \frac{1}{v_0^3} + \frac{p}{u_0^2 v_0^2}$$

- Linear stability analysis

$$u(t) = u_0 + \delta u(t),\, v(t) = v_0 + \delta v(t)$$

$$\ddot{\delta u} + \delta u \left(1 + \frac{3}{u_0^4} + \frac{3p}{u_0^4 v_0}\right) + \delta v \frac{p}{u_0^3 v_0^2} = 0$$

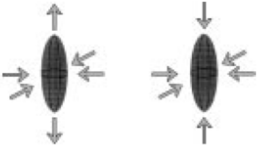
$$\ddot{\delta v} + \delta v \left(\lambda^2 + \frac{3}{v_0^4} + \frac{2p}{u_0^2 v_0^3}\right) + \delta u \frac{2p}{u_0^3 v_0^2} = 0$$

Gaussian approximation (3)

- Previous system of equations can be decoupled by a linear transformation
- As a result, we have frequencies of two low-lying modes - quadrupole mode ω_{Q0} and breathing mode ω_{B0}

$$\omega_{B0,Q0} = \sqrt{2}\left[\left(1+\lambda^2 - \frac{p}{4u_0^2 v_0^3}\right) \pm \sqrt{\left(1-\lambda^2+\frac{p}{4u_0^2 v_0^3}\right)^2 + 8\left(\frac{p}{4u_0^3 v_0^2}\right)^2}\right]$$

Quadrupole mode ω_{Q0}



Breathing mode ω_{B0}

- $p = 15$, $\lambda = 0.021$, $\omega_{Q0} = 2\pi \times 8.2\,\text{Hz}$, $\omega_{B0} = 2\pi \times 462\,\text{Hz}$

Gaussian approximation (4)

- Due to the nonlinear form of the underlying GP equation, we have nonlinearity induced shifts in the frequencies of low-lying modes (beyond linear response)
- Our aim is to describe collective modes induced by harmonic modulation of interaction

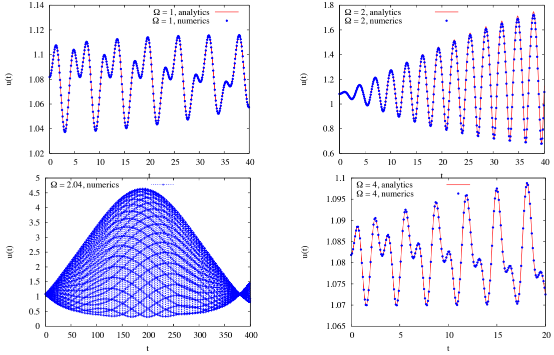
$$p(t) \simeq p + q \cos \Omega t$$

- q - modulation amplitude, Ω - modulation frequency
- For Ω close to some BEC eigenmode we expect resonances - large amplitude oscillations and role of nonlinear terms becomes crucial

Spherical BEC - Condensate dynamics (1)

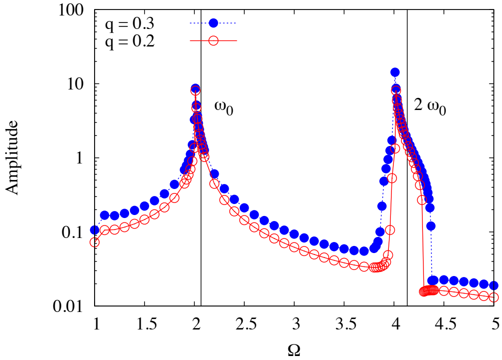
$$\frac{d^2u(t)}{dt^2} + u(t) - \frac{1}{u(t)^3} - \frac{p}{u(t)^4} - \frac{q}{u(t)^4}\cos\Omega t = 0$$

- $p = 0.4$, $q = 0.1$, $u(0) = u_0$, $\dot{u}(0) = 0$
- Linear stability analysis: $\omega_0 = 2.06638$
- Dynamics depends on Ω



Condensate dynamics (2)

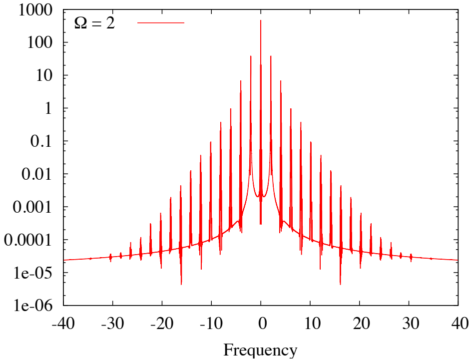
- Resonant behaviour for $\Omega \sim \omega_0$ and $\Omega \sim 2\omega_0$



- Clearly, collective modes are shifted

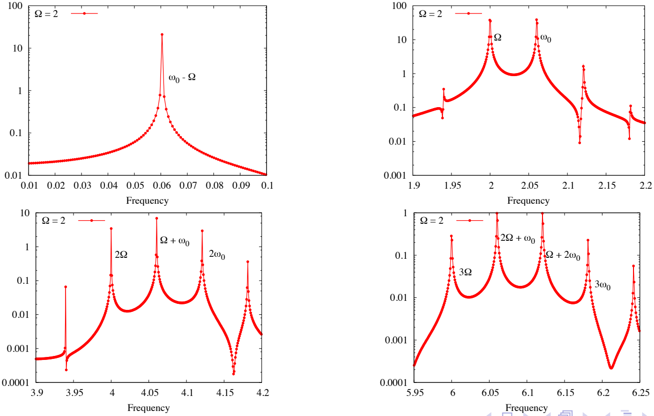
Excitation spectra (1)

- We look at the Fourier transform of $u(t)$, $p = 0.4$, $q = 0.1$ and $\Omega = 2$



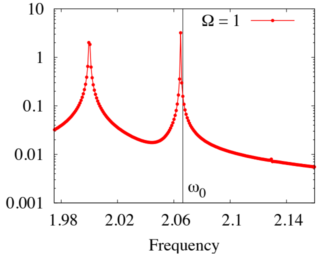
Excitation spectra (2)

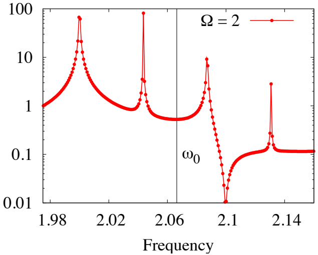
- We have two basic modes Ω and ω_0 and many higher-order harmonics



Excitation spectra (3)

- Frequency of the breathing mode is significantly shifted in the resonant region





Perturbative approach - method

- Linearization of the variational equation yields for vanishing driving $q = 0$ zeroth order collective mode $\omega = \omega_0$ of oscillations around the time-independent solution u_0:

$$\omega_0 = \sqrt{1 + \frac{3}{u_0^4} + \frac{4p}{u_0^5}}\,, \qquad u_0 - \frac{p}{u_0^4} - \frac{1}{u_0^3} = 0$$

- To calculate the collective mode to higher orders, we rescale time as $s = \omega t$:

$$\omega^2\, \ddot{u}(s) + u(s) - \frac{1}{u(s)^3} - \frac{p}{u(s)^4} - \frac{q}{u(s)^4}\cos\frac{\Omega s}{\omega} = 0$$

- We assume the following perturbative expansions in q:

$$\begin{aligned} u(s) &= u_0 + q\, u_1(s) + q^2\, u_2(s) + q^3\, u_3(s) + \ldots \\ \omega &= \omega_0 + q\, \omega_1 + q^2\, \omega_2 + q^3\, \omega_3 + \ldots \end{aligned}$$

Perturbative approach - method

- This leads to a hierarchical system of equations:

$$\begin{aligned}
\omega_0^2\,\ddot{u}_1(s) + \omega_0^2\,u_1(s) &= \frac{1}{u_0^4}\sin\frac{\Omega s}{\omega} \\
\omega_0^2\,\ddot{u}_2(s) + \omega_0^2\,u_2(s) &= -2\omega_0\,\omega_1\,\ddot{u}_1(s) - \frac{4}{u_0^5}\,u_1(s)\,\sin\frac{\Omega s}{\omega} + \alpha\,u_1(s)^2 \\
\omega_0^2\,\ddot{u}_3(s) + \omega_0^2\,u_3(s) &= -2\omega_0\,\omega_2\,\ddot{u}_1(s) - 2\beta\,u_1(s)^3 + 2\alpha\,u_1(s)u_2(s) - \omega_1^2\,\ddot{u}_1(s) \\
&\quad + \frac{10}{u_0^6}\,u_1(s)^2\,\sin\frac{\Omega s}{\omega} - \frac{4}{u_0^5}\,u_2(s)\,\sin\frac{\Omega s}{\omega} - 2\omega_0\,\omega_1\,\ddot{u}_2(s)
\end{aligned}$$

where $\alpha = 10p/u_0^6 + 6/u_0^5$ and $\beta = 10p/u_0^7 + 5/u_0^6$.

- We determine ω_1 and ω_2 by imposing cancellation of secular terms - Poincaré-Lindstedt method

Perturbative approach - method

- Secular term - explanation

$$\ddot{x}(t) + \omega^2 x(t) + C\cos(\omega t) = 0$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t) - \underbrace{\frac{C}{2\omega} t\sin(\omega t)}_{\text{linear in } t}$$

- In order to have properly behaved perturbative expansion, we impose cancellation of secular terms by appropriately adjusting ω_1 and ω_2
- Another way of reasoning

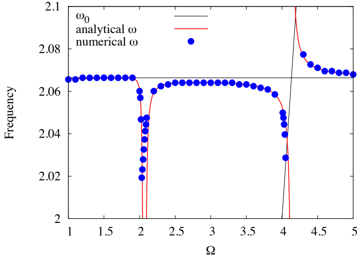
$$u(t) = A\cos\omega t + A_1 t\sin\omega t \approx A\cos\omega t\cos\Delta\omega t + \frac{A_1}{\Delta\omega}\sin\Delta\omega t\sin\omega t$$

$$u(t) \approx A\cos[(\omega - \Delta\omega)t], \quad \Delta\omega = \frac{A_1}{A}$$

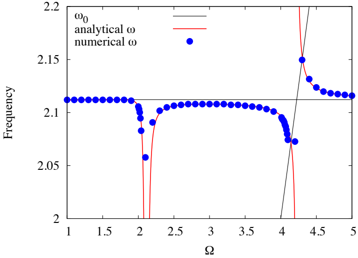
Perturbative approach - results

- Frequency of the breathing mode vs. driving frequency Ω
- Result in second order of perturbation theory

$$\omega = \omega_0 + q^2 \, \frac{\text{Polynomial}(\Omega)}{(\Omega^2 - \omega_0^2)^2 \, (\Omega^2 - 4\omega_0^2)}$$



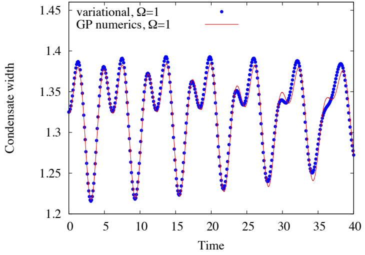
$p = 0.4, \quad q = 0.1$

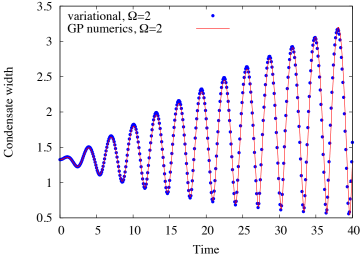


$p = 1, \quad q = 0.2$

GP analysis (1)

- Comparison of the solution of time-dependent GP equation with solution obtained using Gaussian approximation, $p = 0.4$, $q = 0.2$

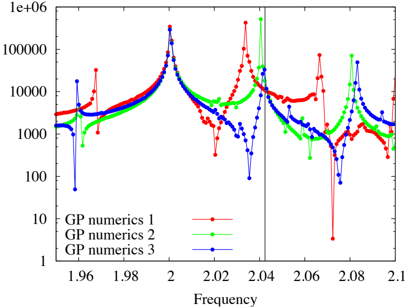




- Good quantitative agreement even for long times

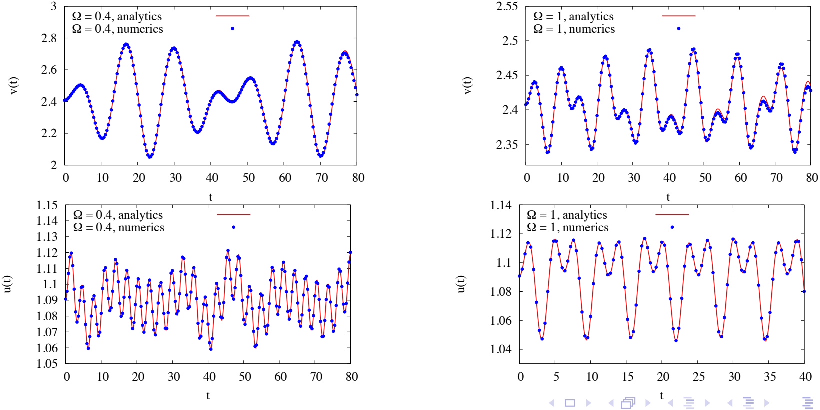
GP analysis (2)

- Comparison becomes even more evident by looking at the Fourier spectrum of the solution of GP equation, $p = 0.4$, $q = 0.2$, $\Omega = 2$



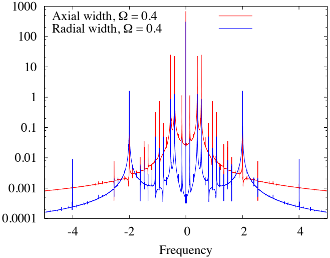
Cylindrical BEC - Condensate dynamics

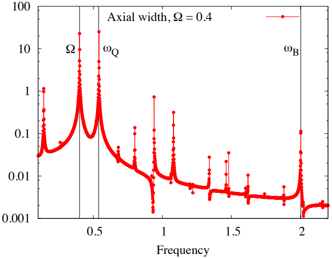
- $p = 1$, $q = 0.2$, $\lambda = 0.3$
- Linear stability analysis: quadrupole mode $\omega_{Q0} = 0.538735$, monopole mode $\omega_{B0} = 2.00238$



Excitation spectra (1)

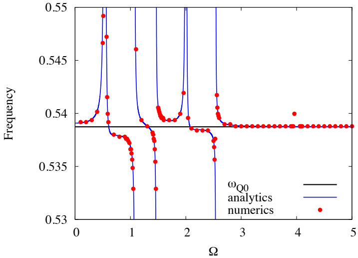
- We have three basic modes: ω_Q, ω_B, Ω and many higher-order harmonics





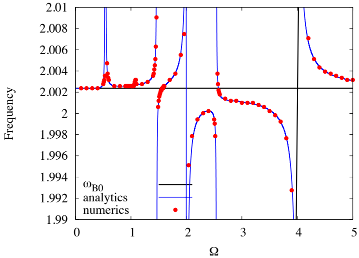
Excitation spectra (2)

- Frequency of quadrupole mode ω_Q versus driving frequency Ω



- Poles: ω_{Q0}, $\omega_{B0} - \omega_{Q0}$, $2\omega_{Q0}$, $\omega_{Q0} + \omega_{B0}$, ω_{B0}

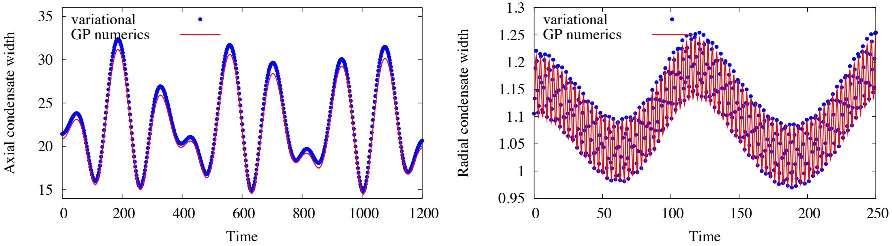
- Frequency of breathing mode ω_B versus driving frequency Ω



- Poles: ω_{Q0}, $\omega_{B0} - \omega_{Q0}$, ω_{B0}, $\omega_{Q0} + \omega_{B0}$, $2\omega_{B0}$

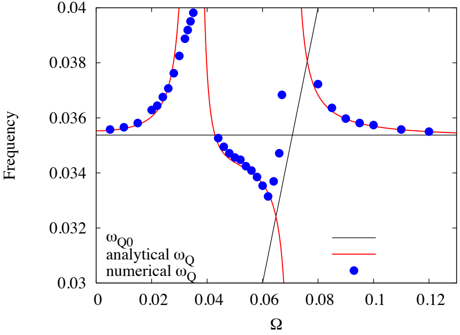
Experimental setup - condensate dynamics

- Comparison of the solution of time-dependent GP equation with Gaussian approximation
  $p = 15$, $q = 10$, $\lambda = 0.021$ and $\Omega = 0.05$



Experimental setup - frequency shifts

- In the experiment:
  - $\omega_B >> \omega_Q$, $\Omega \in (0, 3\omega_Q)$, large modulation amplitude
  - Strong excitation of quadrupole mode
  - Excitation of breathing mode in the radial direction
  - Frequency shifts of quadrupole mode of about 10% are present

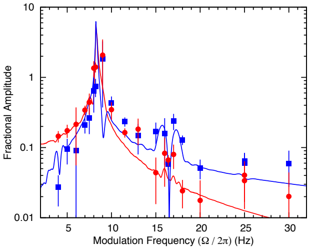


Experimental setup - experimental analysis

- Resonance curve from PRA **81**, 053627 (2010) has been obtained by a very simplified approach
- Experimental data were fit to the linear combination of two basic harmonics

$$v(t) = v_0 + v_\Omega \sin(\Omega t + \Phi) + v_Q \sin(\omega_{Q0} t + \phi)$$

- Higher harmonics were neglected completely
- Frequency shifts were not included in the analysis
- More careful analysis of experimental data is necessary



Conclusions

- Motivated by recent experimental results, we have studied nonlinear BEC dynamics induced by harmonically modulated interaction
- We have used a combination of an analytic perturbative approach, numerics based on Gaussian approximation and numerics based on full time-dependent GP equation
- Relevant excitation spectra have been presented and prominent nonlinear features have been found: mode coupling, higher harmonics generation and significant shifts in the frequencies of collective modes
- Our results are relevant for future experimental designs that will include mixtures of cold gases and their dynamical response to harmonically modulated interactions

Outlook

- Parametric stabilization of attractive BEC
- Confinement induced resonance
  - We try to excite quadrupole mode only

$$\vec{u}(t) = \begin{pmatrix} u_\rho(t) \\ u_z(t) \end{pmatrix}, \quad \vec{u}(0) = \vec{u}_{\text{eq}} + \epsilon \vec{u}_{Q0}, \quad \dot{\vec{u}}(0) = \vec{0}$$

  - Nonlinearity leads to the coupling of quadrupole and breathing mode
  - This coupling is particulary strong for certain values of trap anisotropy λ
  - Signicant frequency shifts in the frequencies of collective modes may appear