Pure state dynamics of open quantum systems and quantum control

Nikola Burić
Institute of Physics, University of Belgrade,
PO BOX 68, 11000 Beograd, Serbia.

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PLAN:

A) **Introduction:**

-Feedback control of a quantum system requires dynamical theory of continuous quantum measurement.

-Continuous measurement $\rightarrow$ restricted path integral $\rightarrow$ complex Hamiltonian $\rightarrow$ QSD stochastic Schroedinger eq.;

- Open quantum systems ; Master equations; Stochastic Schroedinger eq.;

-SDDE for the feedback control of a quantum system

B) **Recent application of QSD on dynamics of qubits:**

-Evolution of entanglement in open systems of interacting qubits

-Dynamics of interacting qubits pair constrained on the separable states; Nonlinear evolution of coarse-grained quantum system.

-Diffusion geometry and stationary state of QSD for qubits.

-Decoherence in a system of qubits interacting with a nonlinear dissipative quantum or classical system.

-Dynamics with continuous measurement of macroscopic observables

-Geometric phase of an open quantum system

Evolution of a continuously observed quantum system is to be described by simultaneous application of the Schroedinger unitary evolution and the projection postulate.

Propagator of the unitary evolution in the path integral form

$$U_t(q'', q') = \int d[q] exp(iS(q)) = \int d[p]d[q]exp(i \int_{t'}^{t''} (p\dot{q} - H(p, q, t)), \tag{1}$$

Integrand is interpreted as the amplitude of the probability that the system was evolving along a particular path with fixed initial and finial points.

An observable A is measured during time (t', t'') with the result $a(t)$. At each moment $t \in (t', t'')$ the system has a sharp value of the observable A that must be in a small interval around $a(t)$ given by the experimental error. The integration in (2) is effectively restricted only over paths in a small cylinder around the path given by $\{A(p,q) = a(t) = a((p(t), q(t)), t \in (t', t'')\}$. Thus

$$U_t^a(q'', q') = \int d[p]d[q]w_a(q,p)exp(i\int_{t'}^{t''}(p\dot{q} - H(p,q,t)), \qquad (2)$$

where $w_a(q,p)$ is zero if $A(p(t), q(t))$ is not close to the monitored values $a(t)$.

We suppose that $w_a(q,p)$ is of the following Gaussian form:

$$w_a(q,p) = exp\left(-k\int_0^T (A-a)^2 dt\right), \qquad (3)$$

where k is related to the measurement accuracy, i.e. if $k = 1/T\Delta a_T^2$ then Δa_T is the error of the measurement that last for the time T.

Substitution in (3) gives

$$U_t^a(q'',q') = \int d[p]d[q]exp\left(\int_{t'}^{t"} i(p\dot{q} - H(p,q,t)) - k\int_0^T (A-a)^2 dt\right) \tag{4}$$

The propagator correspond to the evolution equation with the complex Hamiltonian $\mathcal{H}_a$.

$$|\psi_t >= -i\mathcal{H}_a|\psi_t >= \left[iH - k(A-a(t))^2\right]|\psi_t > . \tag{5}$$

This equation describes evolution due to the system's internal interactions combined with the selective continuous measurement of an observable A with the recorded result $a(t)$.

The pure state $|\psi_t >$ that satisfies (5) is not normalized. Introduce a function $\xi(t)$ and $dw = \xi dt$ such that the norm of

$$|\Psi_t >= exp\left(\frac{1}{2}\int_0^t dt\xi(t)^2\right)\psi(t) \tag{6}$$

is conserved. Introduce new notation: $c(t) = a(t) + \xi(t)/\sqrt(2k)$.

With the new notation equation (5) reads

$$d|\Psi >= \left[-iH - k(A-c)^2\right]|\Psi > dt + \sqrt(2k)(A-c)|\Psi > dw. \tag{7}$$

Conservation of the norm $< \Psi + d\Psi|\Psi + d\Psi >=< \Psi|\Psi >$ is satisfied if

$$c =< \Psi|A|\Psi > \qquad \text{and} \qquad dw^2 = dt \tag{8}$$

The function $\xi(t)$ is the white noise process so that the process $\Psi(t)$ represent a diffusion in the Hilbert space of the system's pure states. Since the Hilbert space is complex, it is natural to consider complex diffusion, i.e. to assume that dw are increments of a complex Wiener process.

Thus, the evolution of the system under continuous selective measurement of the variable A, with $a(t)$ as the measurement readout, is given by the solutions of

$$d|\psi >= \left[-iH - k(A- < A >)^2\right] |\Psi > dt + \sqrt(2k)(A- < A >)|\Psi > dw, \tag{9}$$

where

$$\begin{aligned} a(t) &= < \Psi|A|\Psi > +\xi(t)/\sqrt(2k), & (10) \\ \mathrm{E}[dw] &= \mathrm{E}[dwdw] = 0, \qquad dwdw^* = dt. & (11) \end{aligned}$$

The evolution of the state $\rho(t) = E[|\Psi(t) >< \Psi(t)|]$ under the non-selective measurement of A is given by the master equation of the form

$$\frac{d\rho(t)}{dt} = -i[H, \rho] - \frac{k}{2}[A[A, \rho]], \tag{12}$$

This is in the form of the Linblad master equation, with one Linblad generator A, for the contniuous semi-group of completely positive maps. The general form of this equation is

$$\frac{d\rho(t)}{dt} = -i[H, \rho] - \frac{1}{2}\sum_k [L_k \rho, L_k^\dagger] + [L_k, \rho L_k^\dagger], \tag{13}$$

where $-i[H, \rho]$ describes the unitary part and the rest is the dissipative part. The equation describes evolution of an open quantum system under the Markov assumption.

There is the corresponding QSD unraveling of the Linblad master equation in the general case. The evolution equation is:

$$
\begin{aligned}
|d\psi> &= -iH|\psi> dt \\
&+ \left[\sum_k 2 < L_k^\dagger > L_k - L_k^\dagger L_k - < L_k^\dagger >< L_k >\right] |\psi(t) > dt \\
&+ \sum_k (L_l - < L_k >)|\psi(t) > dW_k
\end{aligned} \tag{14}
$$

where $<>$ denotes the quantum expectation in the state $|\psi(t)>$ and dW_k are independent increments of complex Wiener c-number processes $W_k(t)$.

- Relation between QSD and Lindblad eq. is unique.

-Wave packets are often localized in small neighb. of the phase space points. Hamiltonian→ dispersion (delocalization) vs. Lindblads (environment) → localization onto minimum uncertainty states.

Feedback control of quantum system

QSD equation in some basis has the typical SDE form:

$$dx(t) = f(x(t))dt + \sum_{k=1}^{m} B_k(x(t))dW_k, \qquad x \in \mathbf{R}^{2n}.$$

-Dynamical eq. of the feedback control is a SDDE:

$$dx(t) = [f(x(t))+g(x(t-\tau))]dt+\sum_{k=1}^{m'} B'_k(x(t), x(t-\tau))dW_k, \qquad c \in \mathbf{R}^{2n}$$

-If for any state ψ_f such that $< \psi_f|A_i|\psi_f >= a_i$ there are g and B'_k such that $\psi(t)$ converges to ψ_f then the system is controllable.

-Unitary and dissipative-stochastic controllers are allowed.

-Use the Lyapunov functional method for SDDE to prove controllability in specific examples and generally.

-Entanglement dynamics in systems of pairwise coupled qubits

Ring lattice with N sites:

$$H = \sum_i^N \vec{\omega}_i \vec{\sigma}_i + \sum_i^N (\vec{\sigma}_i) J^{i,i+1} (\vec{\sigma}_{i+1}), \quad N+1=1 \tag{15}$$

QUESTION: Is there a general relation between entanglement evolution and qualitative properties of the system dynamics (with environments)?

Measures of local or global entanglement can be efficiently calculated using the reduced density matrices ρ_{ij}. These are determined by correlations between different components $Tr[\rho\sigma_i^j\sigma_{i+1}^k]$; $i = 1,\dots N$, $j,k = x,y,z$ which are calculated as stochastic averages $E[< \sigma_i^j \sigma_{i+1}^k >]$ over realizations of the stochastic process.

Examples:

-Heisenberg interaction in fixed external field along z-axis

$$H = \sum_i^N \omega\sigma_i^z + J\sum_i^N (\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z). \tag{16}$$

- Transverse Izing interaction: $H = \sum_i^N \omega\sigma_i^z + J\sum_i^N \sigma_i^x\sigma_{i+1}^x$

- XZ models

$$H(\omega_z, \omega_x, J) = \sum_i^N \omega_z\sigma_i^z + \sum_i^N \omega_x\sigma_i^x + J\sum_i^N \sigma_i^x\sigma_{i+1}^x. \tag{17}$$

-Heisenberg (symmetric) vs transverse Izing (nonsymetric)(NB Phys.Rev. 2006,2008).

- $\omega_z\omega_x = 0$ (Q-integrable) vs $\omega_z\omega_x \neq 0$(Q-nonintegrable Q-chaotic)(example of JJ-qubits as the Q-chaotic in NB, Phys. Rev.A 2005; general case in NB Phys.Lett A 2009).

Examples of Linblads (environments) for the system of N qubits are:

For the dephasing env.

$$L_i = \mu \sigma_i^+ \sigma_i^- . \tag{18}$$

For the thermal env.

$$L_i = \frac{\Gamma(\bar{n}+1)}{2} \sigma_i^- + \frac{\Gamma \bar{n}}{2} \sigma_i^+ . \tag{19}$$

Measurement of $\sigma_{x,y,x}$: Hermitian $L = \sigma_{x,y,z}$.

Initial states

The separable states

$$|\mathrm{sep}> \equiv |\rightarrow_1, \uparrow_2, \uparrow, \ldots \uparrow_N>, \tag{20}$$

the states with only one $(i, i+1)$-pair maximally entangled and the rest in product form

$$|\mathrm{max}> \equiv [(|\uparrow_1, \downarrow_2> + |\downarrow_1, \uparrow_2>) \otimes |\downarrow_3, \ldots \downarrow_N)]/\sqrt{2} \tag{21}$$

and an example of a state with distributed entanglement

$$\begin{aligned} |\mathrm{W}> \quad &\equiv \quad (|\uparrow_1, \downarrow_2, \downarrow_3 \ldots \downarrow_N> + |\downarrow_1, \uparrow_2, \downarrow_3 \ldots \downarrow_N> \ldots \\ &+ \quad |\downarrow_1, \downarrow_2 \ldots \uparrow_N>)/\sqrt{N}. \end{aligned} \tag{22}$$

Geometry of diffusion and stable states

The complex n-dimensional equation

$$dc = f(c)dt + \sum_{k=1}^{m} B_k(c)dW_k, \qquad c \in \mathbf{C}^n \tag{23}$$

generates 2n-dimensional real diffusion (eq. (27)). Introduce real n dimensional vectors

$$p = \frac{i}{\sqrt{2}}(\bar{\psi} - \psi), \qquad q = \frac{1}{\sqrt{2}}(\bar{\psi} + \psi), \tag{24}$$

and a $2n$ dimensional vector $X = (q, p)$. Similarly, introduce real and imaginary parts of the vector f and order them as components of a 2n real vector $\mathcal{F} = (f^R, f^I)$, and introduce real and imaginary parts of the increments of the complex m-dim Wiener process dW by

$$dW_i = (dW_i^R + idW_i^I)/\sqrt{2},\ i = 1, 2, \ldots m \tag{25}$$

Substitution of the complex equation and its complex conjugate, leads to the following $2n$ dimensional real SDE:

$$\begin{pmatrix} dq \\ dp \end{pmatrix} = \begin{pmatrix} f^R(p,g) \\ f^I(p,q) \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} B^R & -B^I \\ B^I & B^R \end{pmatrix} \begin{pmatrix} dW^R \\ dW^I \end{pmatrix}, \tag{26}$$

(Phase space picture of QSD; NB J.Phys.A 2005).

The matrix $\mathcal{B}$ of dimension $2n \times 2m$

$$\mathcal{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} B^R & -B^I \\ B^I & B^R \end{pmatrix}, \tag{27}$$

where

$$(B)_{ij} = (B^R)_{ij} + i(B^I)_{ij} \tag{28}$$

gives the diffusion matrix $\mathcal{G}$ for the real $2n$ dimensional diffusion described by the process $\mathcal{G} = \mathcal{B}\mathcal{B}^T$. The matrix $\mathrm{Diag}\{1/2, 1/2, \ldots, 1/2\} + \mathcal{G}$ gives a Riemannian metric on the real $2n$ dimensional vector space.

States with maximal negative/positive curvature are asymptotically unstable/stable.[NB, J.Phys.A 2007]

Nonlinear evolution of two qubits constrained on the manifold of states with no entanglement displays chaotic dynamics typical of (classical) Hamiltonian systems.

Two qbits with the following Hamiltonians

$$H_s = \omega\sigma_z \otimes \mathbf{1} + \omega\mathbf{1} \otimes \sigma_z + \mu\sigma_z \otimes \sigma_z$$

and

$$H_{ns} = \omega\sigma_z \otimes \mathbf{1} + \omega\mathbf{1} \otimes \sigma_z + \mu\sigma_x \otimes \sigma_x$$

both generate integrable Hamiltonian dynamical systems

$$\frac{dx^l}{dt} = 2\Omega^{l,k}\nabla_k H(x),$$

on the phase space S^7/S^1.

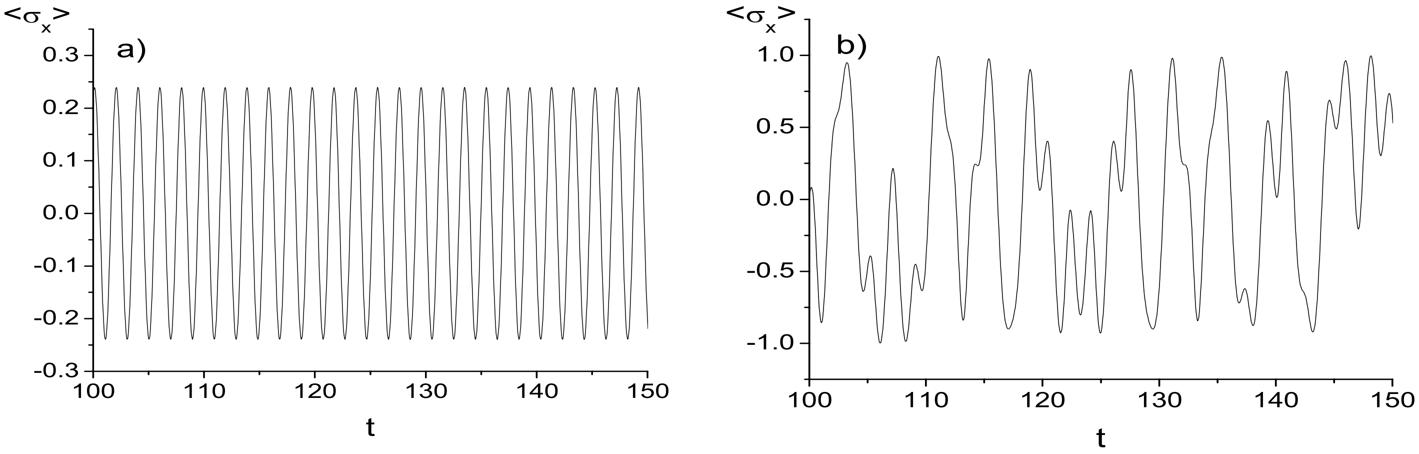
Submanifold of separable states is characterized by two real constraints $f_1 = 0 = f_2$ equivalent to the equation

$$c_1 c_4 - c_2 c_3 = 0$$

Dynamics constrained on this submanifold

$$\dot{X} = \Omega(\nabla X, \nabla H'), \qquad H' = H + \sum_j^k \lambda_j f_j$$

is integrable if the Hamiltonian is symmetric and nonintegrable (chaotic) if H is not symmetric.



Coarse-graining by local observables $\sigma_{x,y,z} \otimes 1; 1 \otimes \sigma_{x,y,z}$ **is enough to generate a dynamical system with typical properties of Hamiltonian chaos.**

Noninteracting qubits in a dissipative nonlinear field

-Hamiltonian H_l and the single Linblad operator L are:

$$H_l = P^2/2 + \beta^2 Q^4/4 - Q^2/2 + g\cos(t)Q/\beta + \gamma(QP + PQ)/2 \quad (29)$$

$$L = \sqrt{2\gamma}a = \sqrt{2\gamma}(Q - iP)/\sqrt{2} \quad (30)$$

- The Hamiltonian of a pair of qubits in an external fixed field reads:

$$H_s = \omega\sigma_x^1 + \omega\sigma_x^2, \quad (31)$$

-The qubits do not interact directly but are both linearly coupled to the Q variable of the large system, i.e. to the Duffing oscillator

$$H_{ls} = \mu Q\sigma_z^1 + \mu Q\sigma_z^2. \quad (32)$$

Notice that the qubits self-hamiltonian H_s and the interaction hamiltonian H_{ls} do not commute.

- Qualitative properties of the evolution of the average values $< Q >, < P >, < \sigma^1 >, < \sigma^2 >$ along an orbit of the QSD equation are determined in different ways by the values of the parameters β, γ, g and μ.

- Decoherence is much more effective when the dissipative oscillator is in the semi-classical rather then in the fully quantum regime. Larger values of the bifurcation parameter g, corresponding to the chaotic dynamics of the oscillator in the semi-classical regime, imply more rapid decrease of the qubits pair entanglement and faster increase of the qubits von Neumann entropy than the smaller values of g corresponding to regular motion. (N.B. Phys.Rev.A 2009, N.B. Phys.Lett (t.a.))

Dynamics with continuous measurement of macroscopic variables

-Consider dynamics of:

$$\sigma_{x,y,z} = \frac{1}{N}\sum_{i}^{N}\sigma^{i}_{x,y,z}$$

for large N, determined by a Hamiltonian H of local or long-range interactions .

The main result is that for systems with qualitatively different but local interaction the macroscopic coarse-graining is enough to induce the dispersionless evolution of the macroscopic variables, while for systems with long range global interaction some form of environmental decoherence is also required.

(N.B. Phys.Rev.A 2009)

Conclusions

-QSD method can be efficiently used to study entanglement dynamics in strongly entangled qubits chains of moderate lengths in realistic conditions.

-There is a relation between the qualitative prop. of entanglement evolution and the symmetry of the quantum system.

-There is a relation between the qualitative prop. of entanglement evolution and the QI and QC of the quantum system (which depends on the properties of the initial state).

-Geometric properties of the (quantum state) diffusion metric, like its curvature, are indicators of the asymptotically stable states.

-In classical mechanics sufficient symmetry is (almost) synonymous with complete integrability, and the lack of symmetry implies classical chaoticity. In the quantum case the relations are more subtle.