Anatomy of fractional quantum Hall states

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Outline :

- 1. FQHE : testing anyonic statistics
- 2. wavefunctions and numerics
- 3. entanglement spectrum
- 4. conformal limit
- 5. Conclusion

FQHE : testing anyonic statistics

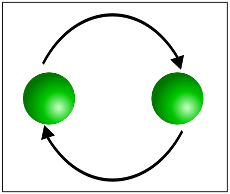
Anyons

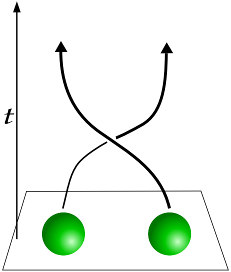
$$\mathcal{P}\Psi\left(r_1, r_2\right) = \Psi\left(r_2, r_1\right) = e^{i\varphi}\Psi\left(r_1, r_2\right) \qquad \mathcal{P}^2\Psi\left(r_1, r_2\right) = \Psi\left(r_1, r_2\right)$$

$\varphi = 0 \ \rightarrow$ bosons or $\varphi = \pi \ \rightarrow$ fermions

experimentally checked !

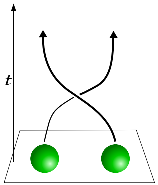
Wilzcek approach : a continuous and adiabatic process





Beyond fermions and bosons in 2D

Toward non-abelian statistics

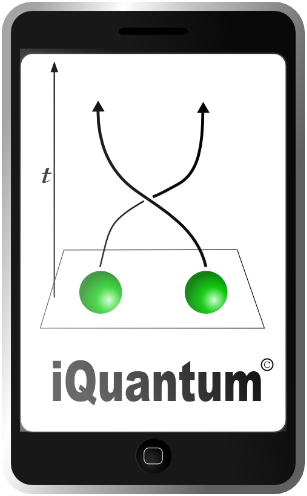
$\Psi_a \longrightarrow$  $\longrightarrow \sum_b U_{ab}\Psi_b$

exchange is described by a (non diagonal)matrix U_{ab}

- swap 1 and 2 : $\Psi_a \to \sum_b U_{ab}^{12}\Psi_b$
- swap 2 and 3 : $\Psi_a \to \sum_b U_{ab}^{23}\Psi_b$

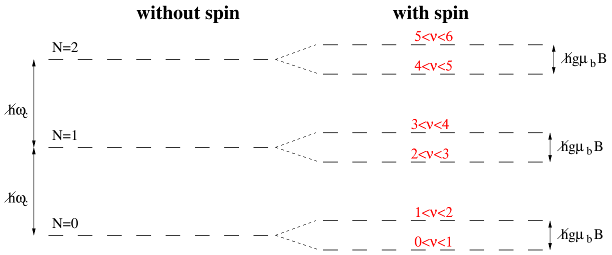
U_{ab}^{12} and U_{ab}^{23} do not commute

FQHE : an experimental test at hand !



Topological quantum computing :
quantum computation in your mobile phone

Landau level



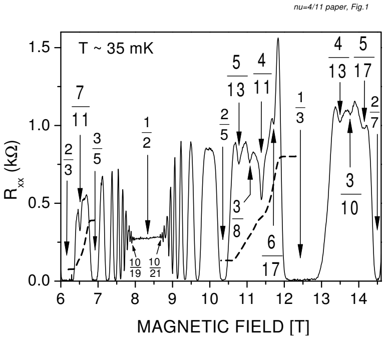
Filling factor : $\nu = \frac{hn}{eB} = \frac{N}{N_\phi}$

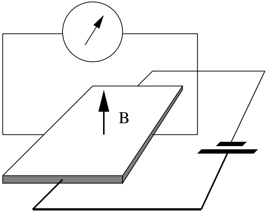
Cyclotron frequency : $\omega_c = \frac{eB}{m}$

Lowest Landau level $(\nu < 1)$: $z^m \exp\left(-|z|^2/4l^2\right)$

N-body wave function : $\Psi = P(z_1, ..., z_N) \exp(-\sum |z_i|^2/4)$

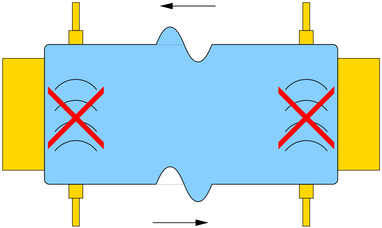
The fractional quantum Hall effect



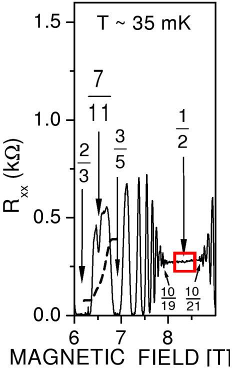


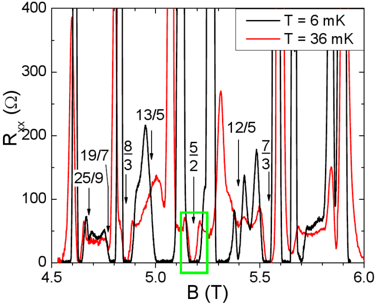
The incompressible liquid picture :

- gap (activated law for $R_{xx}(T)$) from a purely interacting system (no kinetic energy)
- only chiral edge excitations (no back-scattering)



The surprising $\nu = 5/2$ case



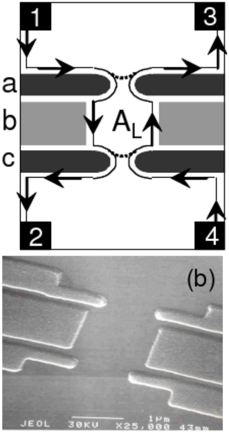


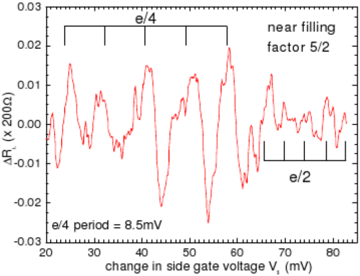
Pan et al, arXiv :0801.1318

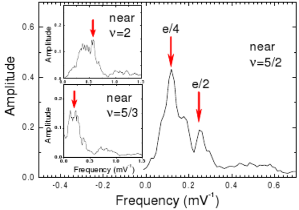
- theory : physics of higher Landau level map onto the LLL physics
- first incompressible state (1987 and 1999) with an even denominator
- theory : Moore-Read state $\longrightarrow$ $e/4$ charges + non abelian excitations

Experimental evidences for non-abelian statistics ?

Interferometer (2 QPC) (*insert lots of theoricists here*)
R.L. Willett, L.N. Pfeiffer, K.W. West **(PNAS 0812599106)**







Another approach : switching noise (Grosfeld, Simon, Stern 06)
experiments by W. Kang (Univ. of Chicago)

Wavefunctions and numerics

The theoretical approaches

FQHE is a hard N-body problem : the Hamlitonian is just the (projected) interaction !

Two major methods :

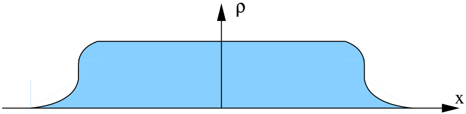
- variational method : find a wave functions describing low energy physics (symmetries, CFT, model...)
- numerical calculation : exact diagonalizations
  - a more realistic description of the physical system
  - the spectrum (complete or partial), the eigenstates, (operator mean values)...
  - require large computer power (dim $\sim 3.10^8 \rightarrow 2.5$Gb per vector)

FQHE can be seen starting from N=4 !

The Laughlin wave function

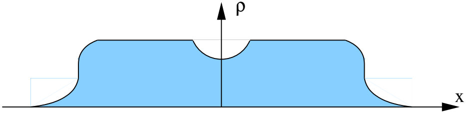
A (very) good approximation of the ground state at $\nu = \frac{1}{3}$

$$\Psi_L(z_1, ...z_N) \quad = \quad \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i \frac{|z_i|^2}{4l^2}}$$



add one flux quantum at z_0 = one quasi-hole

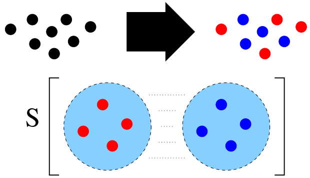
$$\Psi_{qh}(z_1, ...z_N) \quad = \quad \prod_i (z_0 - z_i) \;\; \Psi_L(z_1, ...z_N)$$



- Locally, create one quasi-hole with fractional charge $\frac{+e}{3}$
- “Wilczek” approach : quasi-holes obey fractional statistics

The Pfaffian / Moore-Read state

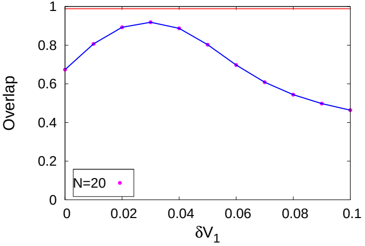
$$\Psi_{pf}(z_1, \ldots, z_N) = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2$$



- correlators of a $\mathbb{Z}_2$ parafermions CFT
- Cappelli et al. (2001) : 2 color approach (bosons)
  - add/remove one flux quanta $\longrightarrow$ create a pair of quasi-holes /quasi-electrons ($\pm e/4$)
  - 2^{n-1} degenerate states for $2n$ quasi-particles $\longrightarrow$ non Abelian statistics !

Pfaffian state : numerical evidences

- incompressible state only for even N (pairing)
- topological properties : sphere (shift between N and N_ϕ) and torus (degeneracies)



Overlap :

- scalar product between a test state and the "exact" state
- require to tune the interaction to get a good agreement
- here 0.919 for $N = 20$, $dim = 1.9 \times 10^8$

When overlaps are misleading

At least one know example where two different states have large overlaps : Abelian (Jain CF) vs non-abelian (Gaffnian).
$N = 16, \nu = 2/5, \mathrm{dim} = 1.5 \times 10^8$, overlap is 0.935.

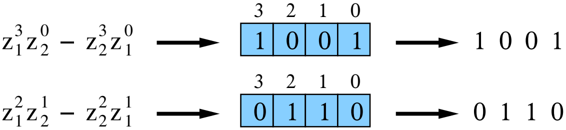
- You can't fight the exponential ! : always a few number of particles $N = 12$ dim=16660(418), $N = 16$ dim=155484150(70180)
- What is the meaning of the overlap ?
  - What is a bad overlap : 0 (wrong quantum numbers) or as good as a random vector $\left(\sim 1/\sqrt{\mathrm{dim}}\right)$
  - What is a good overlap : use Laughlin as a reference ? 0.988 for $\nu = 1/3$ with $\mathrm{dim} = 1.3 \times 10^8$
  - What is missing : how overlap should scale (decrease) with N while preserving the same properties ? Power vs exponential decay ?

Entanglement spectrum

Notations in the LLL

- one-body wavefunction :
  $\phi_m(z) = (2\pi m! 2^m)^{-1/2} z^m \exp(-|z|^2/4)$
- N-body wave functions : $\Psi = P(z_1, ..., z_N) \exp(-\sum |z_i|^2/4)$
- decomposition on the N-body basis $\Psi = \sum_\mu c_\mu sl_\mu$
- Slater determinants sl_μ are labeled by occupation numbers :

$$(z_1 - z_2)^3 \quad = \quad (z_1^3 - z_2^3) - 3(z_1^2 z_2 - z_1 z_2^2)$$



No known formula for the c_μ (even for the Laughlin state !)

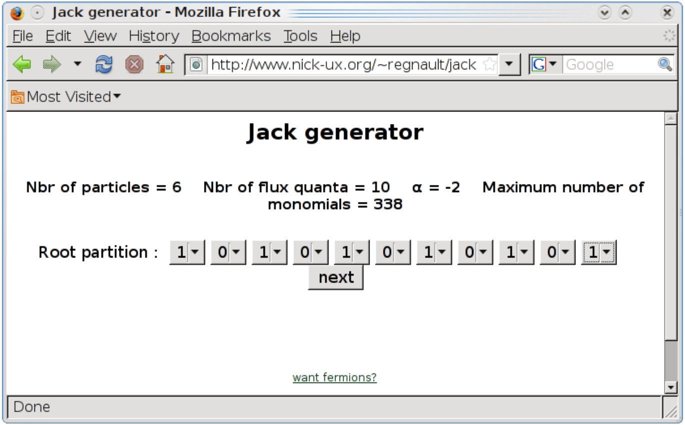
Generating the decomposition

- Mathematica/Maple/Maxima/PhD student :
  analytical calculations (a few particles)
- Exact diagonalizations : directly work in the n-body basis
  both CPU and memory intensive calculations $\Rightarrow$ require good workstation or cluster/supercomputer

Recursion formula for the c_μ for Jack polynomial (B.A. Bernevig and NR)

- for bosons : **Ha (1995), Dumitriua and Shumance (2007)**
- for fermions : brute force way is not trivial (Kostka numbers, Schur polynomials)
- less CPU / memory intensive ($N = 15$, 500 times less CPU time, only a single PC !)
- a few additional system sizes (can't fight the exponential)

Spam



Get your own Jack from the web ! (up to some decent sizes) + entanglement spectrums

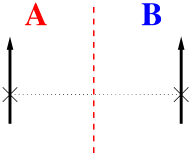
Decomposition vs overlap

- marketing department : decomposition is the state DNA
- mathematics department : this is a basis ! Completely defines the state.
- mathematician vs physicist :
  - two states are equal if they have the same decomposition
  - two states correspond to the same phase, if for large N, all the important measurable quantities are identical.
- much more information than the overlap (billions vs single number)

how to process this huge amount of information ?

Entanglement entropy

example : system made of two spins 1/2



Von Neumann entropy for the pure system

$$\rho = |\Psi\rangle \langle\Psi| \qquad S = -\mathrm{Tr}\,(\rho \log \rho) = 0$$

Reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$

Entropy for the A subsystem ?

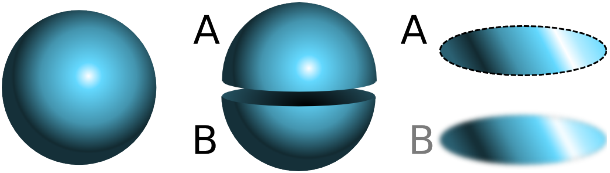
$$|\uparrow\uparrow\rangle \quad \longrightarrow \rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \longrightarrow S_A = 0$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \quad \longrightarrow \rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \longrightarrow S_A = \log 2$$

mesurement of the entanglement

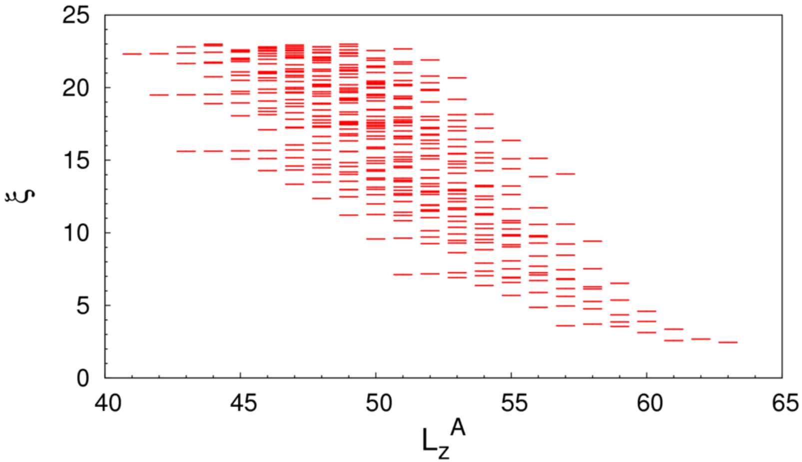
Entanglement entropy for the FQHE

- look at one ground state $|\Psi\rangle$ on the sphere
- cut the system into two parts A and B in orbital space ($\simeq$ real space, geometrical partition)
- reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle \langle\Psi|$, block-diagonal wrt N^A and L_z^A
- compute the entanglement entropy $S_A = -\mathrm{Tr}_A (\rho_A \log \rho_A)$.



- topological entanglement entropy : extract the γ from $S_A = cL - \gamma$ (Haque et al.) : highly non-trivial, require two thermodynamical extrapolations...
- looking at the entanglement spectrum : plot $\xi = -\log \lambda_A$ vs L_z^A for fixed cut and N^A

Entanglement spectrum (Lee and Haldane)

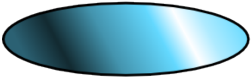


Laughlin $N = 13, l_A = 36$ (hemisphere cut), $N_A = 6$

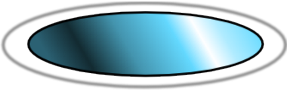
L_z^A angular momentum of A, $\xi = -\log \lambda_A$, λ_A's are ρ_A eigenvalues.

Entanglement spectrum

- Schmidt decomposition $|\Psi\rangle = \sum_p \exp(-\xi/2)\,|A,p\rangle \otimes |B,p\rangle$
- a way to look at the Fock space decomposition
- "banana" shaped spectrum for pure CFT state (not only Jacks) with a given maximum L_z^A
- "low energy" part : a signature of the state (edge mode degeneracy).
- example Laughlin (1,1,2) : Ψ_L, $\Psi_L \times \sum_i z_i$, $\Psi_L \times \sum_i z_i^2$ and $\Psi_L \times \sum_{i<j} z_i z_j$

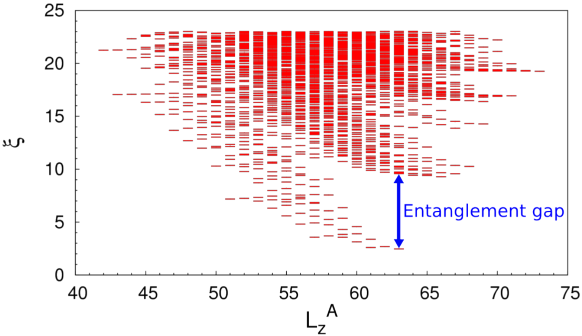


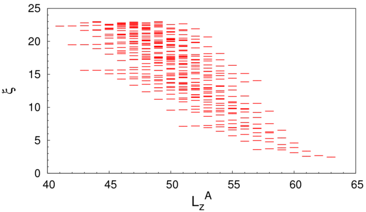
$L^A_{z\,max}$

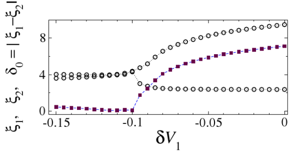


$L^A_{z\,max}$ -1

Coulomb case and entanglement gap



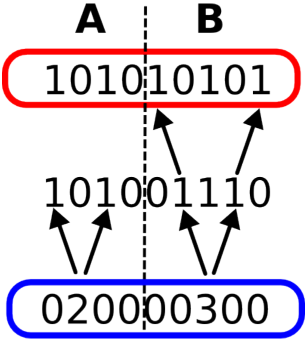




Entanglement spectrum : some results

- probing non abelian statistics **(Lee, Haldane 2008)**
- looking at (precursor of) phase transition through closing entanglement gap **(Zozulya, Haque, NR, 2009)**
- differentiate states with large overlap but different excitations (from the ground state only !) **(NR, Bernervig, Haldane 2009)**
- non-trivial relation between ES and edge mode **(Bernervig, NR 2009)**
- when $N \rightarrow \infty$ recover degenerate multiplets and linear (relativistic) dispersion relation for the edge mode **(Thomale, Stedyniak, NR, Bernervig 2009)**

Disconnected (reducible) squeezing sequences



- a given partition (blue box)
- cut the system into two parts A and B in orbital space ($\simeq$ real space)
- disconnected squeezing sequence if you can reach the root partition (red box) without involving squeezing between A and B

- Product rule : $c_{020000300} = c_{0200} \times c_{00300}$
- true for any (fermionic) Jack polynomials !
- unnormalized basis, proof based on the recursion formula

Entanglement spectrum

Why only one state at $L^A_{z\,max}$?

- $L^A_{z\,max}$ is the L^A_z of the root partition
- fixed L^A_z involves partitions with disconnected squeezing sequence between A and B
- use the product rules : $N = 4$ four partitions, half cut

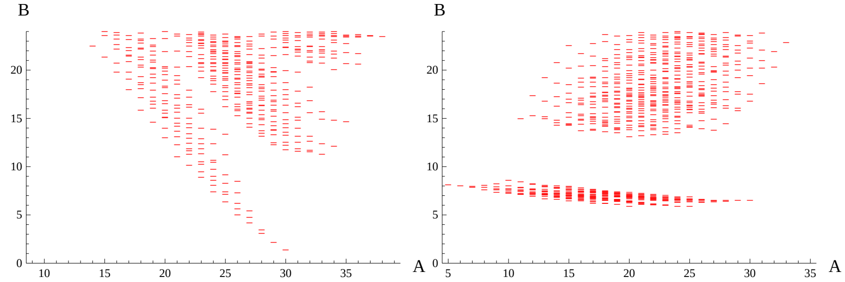
$$\begin{aligned} & c_{1010101}|1010101\rangle \\ + \quad & c_{1010020}|1010020\rangle \\ + \quad & c_{0200101}|0200101\rangle \\ + \quad & c_{0200020}|0200020\rangle \\ = \quad & (c_{1010}|1010\rangle + c_{0200}|0200\rangle) \otimes (c_{101}|101\rangle + c_{020}|020\rangle) \end{aligned}$$

Laughlin liquid in A $\otimes$ Laughlin liquid in B !

Conformal limit

Defining a "clear" entropy gap

- entanglement gap collapses a few momenta away from the maximum one (the system "feels" the edge)
- current definition of the ES contains the magnetic length
- remove the information coming from the geometry
- example : Coulomb $\nu = 1/2$ N=11 bosons



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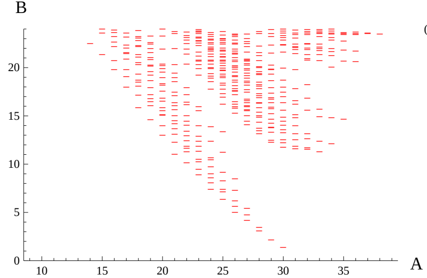
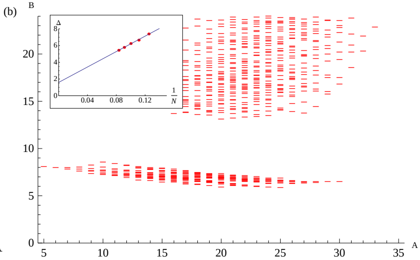
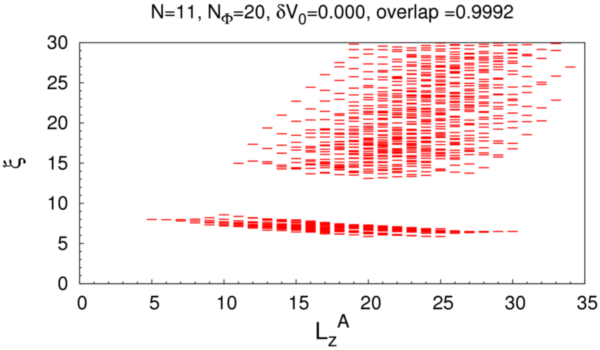




Going through a phase transition

N=11 bosons, $\nu = 1/2$ Coulomb with tuned short range component δV_0

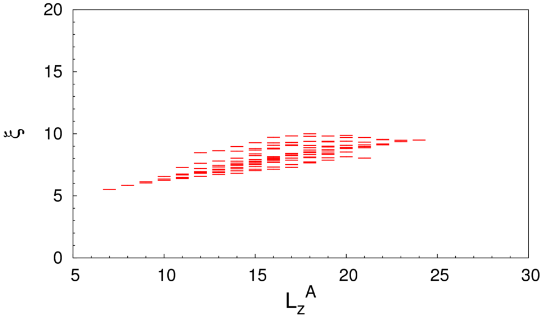


gap closing despite large square overlap (0.989) !

Entanglement adiabatically continuable states

from Moore-Read state to delta ground state N=14 bosons, $\nu = 1$

$$\mathcal{H}_\lambda = (1-\lambda) \sum_{i<j<k} \delta(r_i - r_j)\delta(r_j - r_k) + \lambda \sum_{i<j} \delta(r_i - r_j)$$



No gap closing despite moderate square overlap (0.887) !

Conclusions

- exotic statistics are an exciting topic
- getting closer to experimental evidences in the FQH regime.
- numerical calculations are a powerful method to probe the FQHE
- more tools are needed to clearly identify (precursor of) phases
- entanglement spectrum a way to investigate this problem
- conformal limit a more robust approach to the entanglement spectra
- what about other interacting n-body problem ?

References

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- N. Regnault, B.A. Bernevig, F.D.M. Haldane, PRL 103, 016801 (2009).
- O. Zozulya, M. Haque, N. Regnault, PRB 79, 045409 (2009).

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- code available at http ://www.nick-ux.org/diagham
- entanglement entropy database http ://www.nick-ux.org/ regnault/entropy
- Jack generator http ://www.nick-ux.org/ regnault/jack

postdoc position available at the ENS/Orsay