# Bose-Einstein Condensation in Random Potentials

Robert Graham and Axel Pelster

UNIVERSITÄT DUISBURG ESSEN

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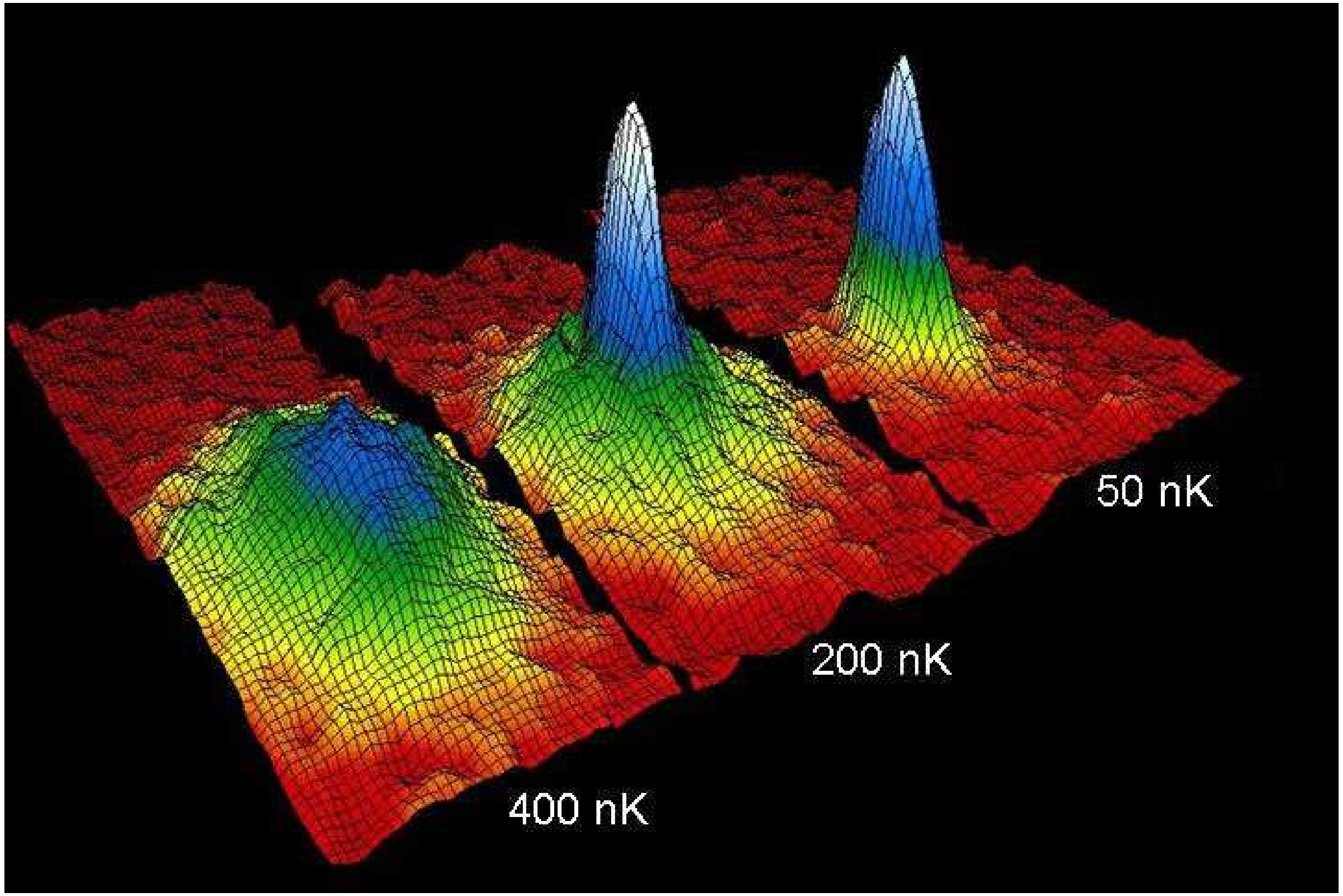
SFB/TR 12: Symmetries and Universality in Mesoscopic Systems

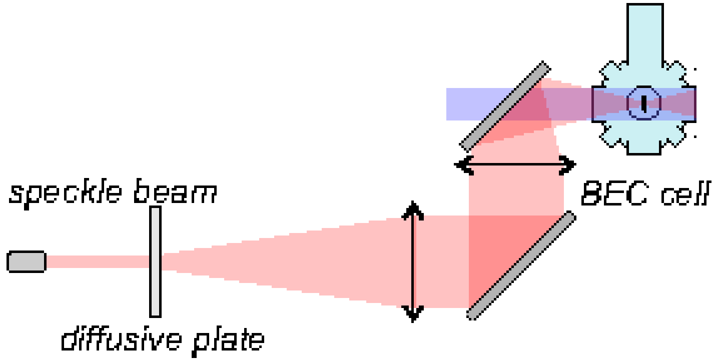
# 1 Introduction: Ultracold Quantum Gases

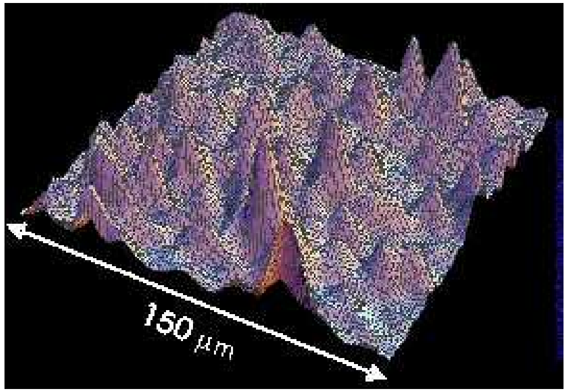


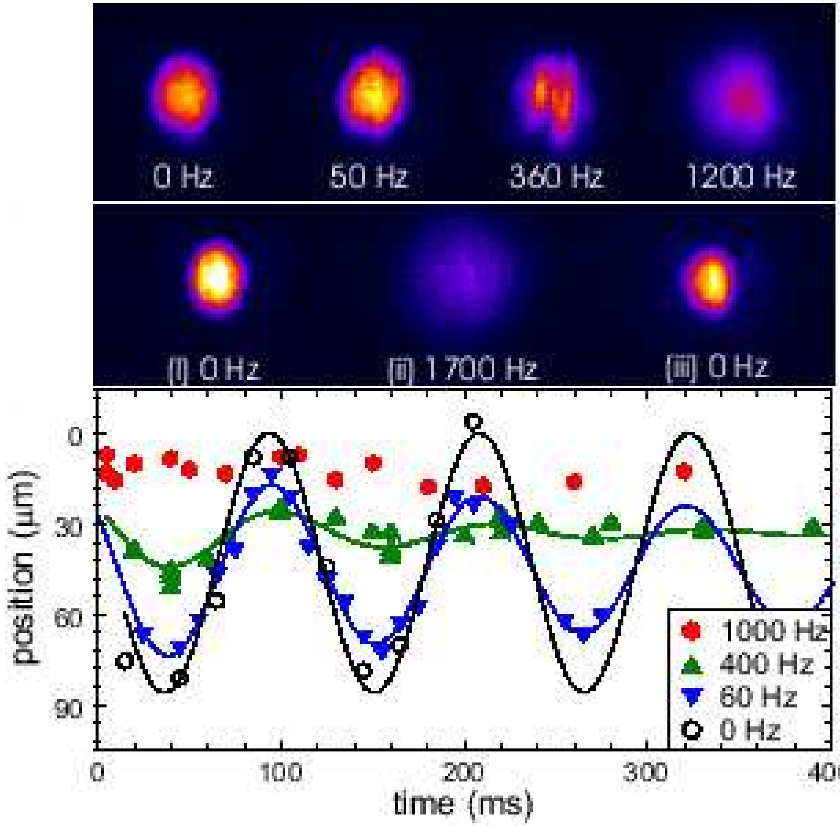

JILA (1995): $^{87}_{37}$Rb , N=20 000 , $\omega_1 = \omega_2 = \omega_3/\sqrt{8} = 2\pi \times$ 120 Hz

# 2.1 Magneto-Optical Trap

## Laser Speckles:



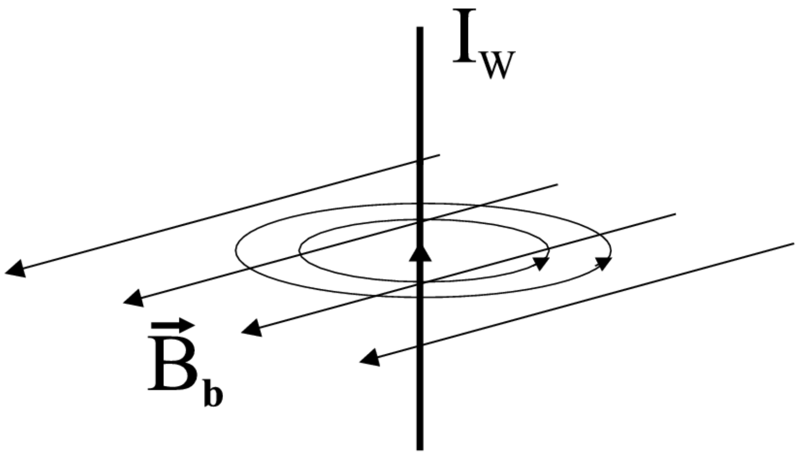



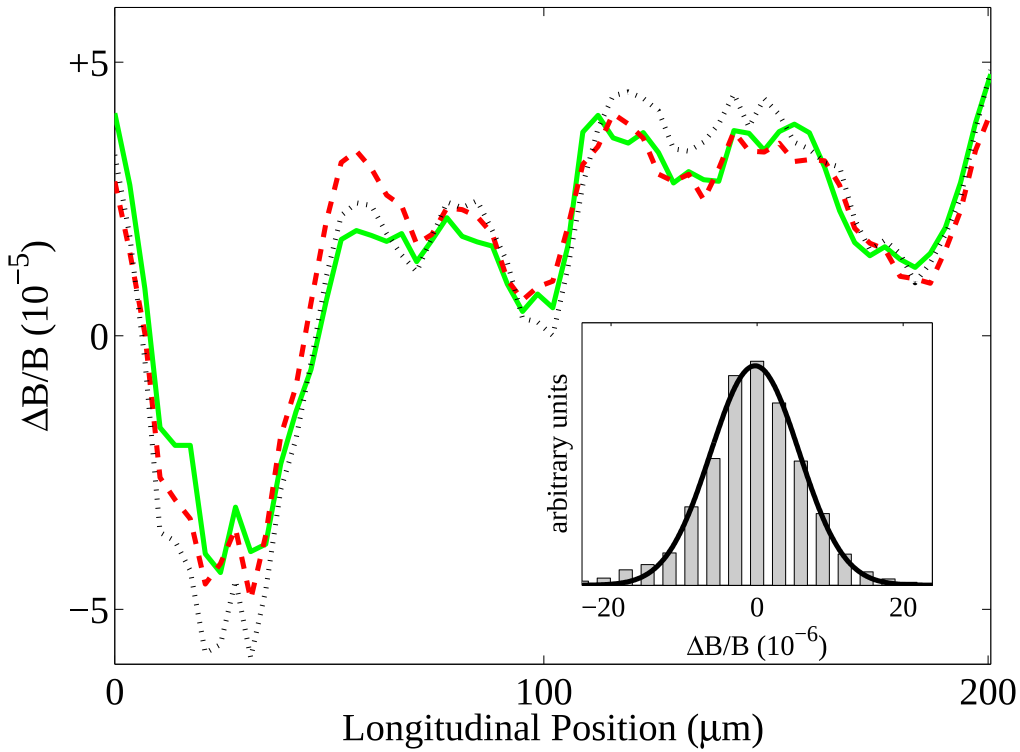




Inguscio *et al.*, PRL **95**, 070401 (2005)

global condensate vanishes

# 2.2 Wire Trap




**Distance:** $d = 10\ \mu$m

**Wire Width:** $100\ \mu$m

**Magnetic Field:** 10 G, 20 G, 30 G

**Deviation:** $\Delta B/B \approx 10^{-4}$

Krüger *et al.*, Phys. Rev. A **76**, 063621 (2007)

Fortàgh and Zimmermann, Rev. Mod. Phys. **79**, 235 (2007)

# 3.1 Model System

## Action of a Bose Gas:

$$\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x \left\{ \psi^* \left[ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \boldsymbol{\Delta} + \boldsymbol{U}(\mathbf{x}) + \boldsymbol{V}(\mathbf{x}) - \boldsymbol{\mu} \right] \psi + \frac{\boldsymbol{g}}{2} \psi^{*2} \psi^2 \right\}$$

## Properties:

- harmonic trap potential: $\boldsymbol{U}(\mathbf{x}) = \frac{M}{2} \omega^2 \mathbf{x}^2$
- disorder potential: $\boldsymbol{V}(\mathbf{x})$ ; bounded from below, i.e. $V(\mathbf{x}) \geq V_0$

$$\overline{V(\mathbf{x}_1)} = 0 \,, \qquad \overline{V(\mathbf{x}_1) V(\mathbf{x}_2)} = R(\mathbf{x}_1 - \mathbf{x}_2) \,, \qquad \ldots$$

- chemical potential: $\boldsymbol{\mu}$
- repulsive interaction: $\boldsymbol{g} = \frac{4\pi\hbar^2 a}{M}$
- periodic Bose fields: $\psi(\mathbf{x}, \tau + \hbar\beta) = \psi(\mathbf{x}, \tau)$

# 3.2 Grand-Canonical Potential

**Aim:**

$$\Omega = -\frac{1}{\beta}\overline{\ln \mathcal{Z}}$$

$$\mathcal{Z} = \oint D\psi D\psi^* \, e^{-\mathcal{A}[\psi^*,\psi]/\hbar}$$

**Problem:**

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$$

**Solution: Replica Trick**

$$\Omega = -\frac{1}{\beta}\lim_{N\to 0}\frac{\overline{\mathcal{Z}^N}-1}{N}$$

# 3.3 Replica Trick

## Disorder Averaged Partition Function:

$$\overline{\mathcal{Z}^N} = \overline{\oint \left\{ \prod_{\alpha=1}^{N} D\psi_\alpha D\psi_\alpha^* \right\} e^{-\sum_{\alpha=1}^{N} \mathcal{A}([\psi_\alpha^*,\psi_\alpha])/\hbar}} = \oint \left\{ \prod_{\alpha=1}^{N} D\psi_\alpha D\psi_\alpha^* \right\} e^{-\mathcal{A}^{(N)}/\hbar}$$

## Replicated Action:

$$\begin{aligned}
\mathcal{A}^{(N)} \quad = \quad & \int_0^{\hbar\beta} d\tau \int d^3x \sum_{\alpha=1}^{N} \left\{ \psi_\alpha^*(\mathbf{x},\tau) \left[ \hbar \frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M} \mathbf{\Delta} + U(\mathbf{x}) - \mu \right] \psi_\alpha(\mathbf{x},\tau) \right. \\
& \left. + \frac{g}{2} \psi_\alpha^*(\mathbf{x},\tau)^2 \psi_\alpha(\mathbf{x},\tau)^2 \right\} \\
& - \frac{1}{2\hbar} \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \int d^3x \int d^3x' \sum_{\alpha=1}^{N} \sum_{\alpha'=1}^{N} \\
& \qquad \times \psi_\alpha^*(\mathbf{x},\tau) \psi_\alpha(\mathbf{x},\tau) R(\mathbf{x}-\mathbf{x}') \psi_{\alpha'}^*(\mathbf{x}',\tau') \psi_{\alpha'}(\mathbf{x}',\tau') + \ldots
\end{aligned}$$

**Similar:** disorder averaged correlation functions

# 4.1 Condensate Density

## Assumptions:

homogeneous Bose gas: $U(\mathbf{x}) = 0$

$\delta$-correlated disorder: $R(\mathbf{x}) = R\,\delta(\mathbf{x})$

## Bogoliubov Theory:

background method: $\psi_\alpha(\mathbf{x},\tau) = \Psi_\alpha + \delta\psi_\alpha(\mathbf{x},\tau)$

replica symmetry: $\Psi_\alpha = \sqrt{n_0}$

**Result:**

$$n_0 = n - \frac{8}{3\sqrt{\pi}}\sqrt{a\,n_0}^{\,3} - \frac{M^2 R}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0}{a}}$$

Huang and Meng, PRL **69**, 644 (1992)

Falco, Pelster, and Graham, Phys. Rev. A **75**, 063619 (2007)

# 4.2 Superfluid Density

## Galilei Boost:

$$\Delta\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x \, \psi^*(\mathbf{x},\tau)\, \mathbf{u}\, \frac{\hbar}{i} \boldsymbol{\nabla}\, \psi(\mathbf{x},\tau)$$

$$d\Omega = -S\, dT - p\, dV - N\, d\mu - \mathbf{p}\, d\mathbf{u}$$

$$\mathbf{p} = - \left.\frac{\partial\Omega(T,V,\mu,\mathbf{u})}{\partial\mathbf{u}}\right|_{T,V,\mu} = MV\, n_n\, \mathbf{u} + \ldots$$

## Result:

$$n_s = n - n_n = n - \frac{4}{3}\frac{M^2 R}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0}{a}}$$

Huang and Meng, PRL **69**, 644 (1992)

Falco, Pelster, and Graham, Phys. Rev. A **75**, 063619 (2007)

# 4.3 Collective Excitations

## Hydrodynamic Equation in Trap With Disorder:

$$m\,\frac{\partial^2}{\partial t^2}\,\delta n(\mathbf{x},t) - \boldsymbol{\nabla}\left[g n_{\rm TF}(\mathbf{x})\boldsymbol{\nabla}\delta n(\mathbf{x},t)\right]$$
$$= -\boldsymbol{\nabla}^2\left[3 g n_R(\mathbf{x})\delta n(\mathbf{x},t)\right] - \boldsymbol{\nabla}\left[\frac{4g}{3} n_R(\mathbf{x})\boldsymbol{\nabla}\delta n(\mathbf{x},t)\right]$$

$n_R(\mathbf{x})$ : Huang-Meng depletion in trap

$n_{\rm TF}(\mathbf{x}) = \left[\mu - V(\mathbf{x})\right]/g$ : Thomas-Fermi density

## Violation of Kohn Theorem:

Surface dipole mode $(n = 0\,, l = 1)$:

$$\frac{\delta\omega_{\rm dip}(\xi = 0)}{\omega_{\rm dip}} = -\frac{5\pi}{16}\,\frac{M^2 R}{8\pi^{3/2}\hbar^4\sqrt{n_{\rm TF}(\mathbf{0})a}}$$

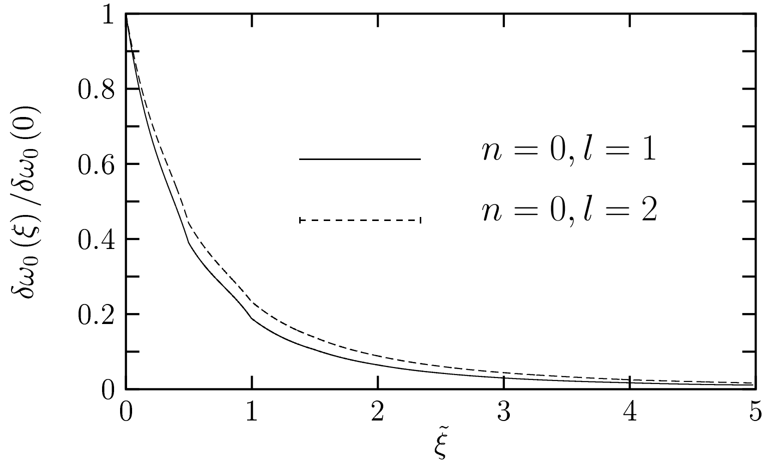
Falco, Pelster, and Graham, Phys. Rev. A **76**, 013624 (2007)

# 4.4 Comparison With Experiment

## Typical Values:

Inguscio *et al.*, PRL **95**, 070401 (2005)

$$\left.\begin{array}{rcl} \xi & = & 10\ \mu\text{m} \\ R_{\text{TF}} & = & 100\ \mu\text{m} \\ l_{\text{HO}} & = & 10\ \mu\text{m} \end{array}\right\} \quad \tilde{\xi} = \frac{\xi R_{\text{TF}}}{l_{\text{HO}}^2 \sqrt{2}} \approx 7$$




$\Longrightarrow$ **Disorder effect vanishes in laser speckle experiment**

## Improvement:

laser speckle setup with correlation length $\xi = 1\ \mu$m

Aspect *et al.*, New J. Phys. **8**, 165 (2006)

$\Longrightarrow$ **Disorder effect should be measurable**

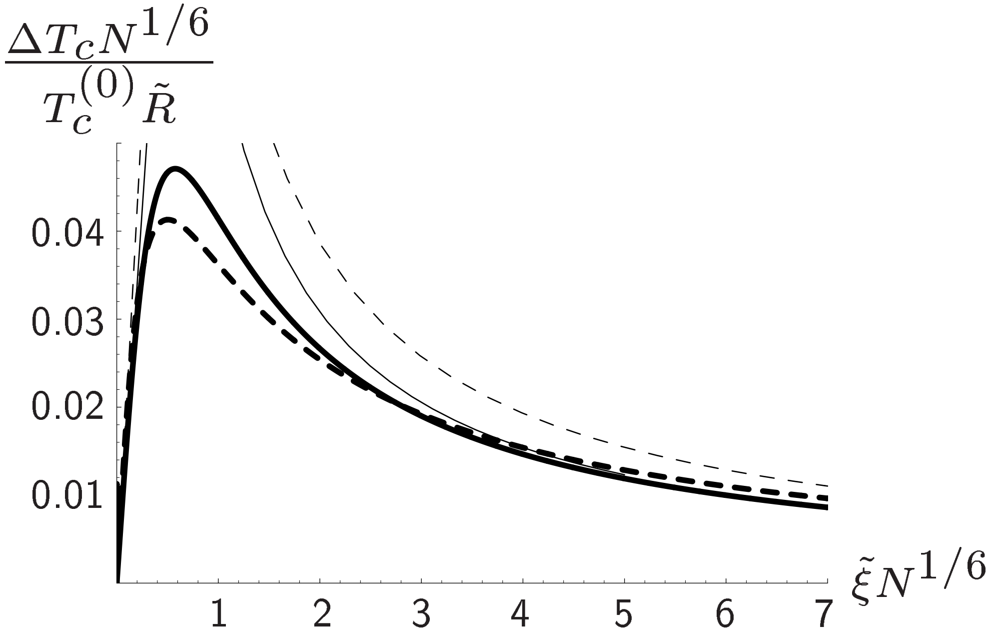
Falco, Pelster, and Graham, Phys. Rev. A **76**, 013624 (2007)

# 5.1 Earlier Results

| trapped Bose gas | homogeneous Bose gas |
|---|---|
| $T_c^{(0)} = \frac{\hbar\omega_{\rm g}}{k_B}\left[\frac{N}{\zeta(3)}\right]^{1/3}$ | $T_c^{(0)} = \frac{2\pi\hbar^2}{k_B M}\left[\frac{n}{\zeta(3/2)}\right]^{2/3}$ |
| $\frac{\Delta T_c}{T_c^{(0)}} = -3.426\,\frac{a}{\lambda_c^{(0)}}$<br>Giorgini et al., PRA **54**, R4633 (1996)<br>Gerbier et al., PRL **92**, 030405 (2004) | $\frac{\Delta T_c}{T_c^{(0)}} = 1.3\,an^{1/3}$<br>Kleinert, Mod. Phys. Lett. B **17**, 1011 (2003)<br>Kastening, Phys. Rev. A **69**, 043613 (2004) |
| $R(\mathbf{x}) = ?$<br>$\frac{\Delta T_c}{T_c^{(0)}} = ?$ | $R(\mathbf{x}) = R\,\delta(\mathbf{x})$<br>$\frac{\Delta T_c}{T_c^{(0)}} = -\frac{M^2 R}{3\pi[\zeta(3/2)]^{2/3}\hbar^2 n^{1/3}}$<br>Lopatin and Vinokur, PRL **88**, 235503 (2002) |

**Procedure:** $n = n(\mu), \quad \mu \nearrow \mu_c \quad \Rightarrow \quad T_c$

# 5.2 Our Results




solid: Gaussian

dashed: Lorentzian

**Length Scale:** $$l_{\rm HO} = \sqrt{\frac{\hbar}{M\omega_{\rm g}}} \,, \qquad \omega_{\rm g} = (\omega_1\omega_2\omega_3)^{1/3}$$

**Dimensionless Units:** $$\tilde{\xi} = \frac{\xi}{l_{\rm HO}} \,, \qquad \tilde{R} = \frac{R}{\left(\frac{\hbar^2}{M l_{\rm HO}^2}\right)^2 l_{\rm HO}^3} = \frac{M^{3/2} R}{\hbar^{7/2}\omega_{\rm g}^{1/2}}$$

Timmer, Pelster, and Graham, Europhys. Lett. **76**, 760 (2006)

Klünder, Pelster, and Graham, to be published

# 6.1 Order Parameters

## Definition:

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau)\rangle} = n_0$$

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \overline{|\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau)\rangle|^2} = (n_0+q)^2$$

## Note:

$q$ is similar to Edwards-Anderson order parameter of spin-glass theory

## Hartree-Fock Mean-Field Theory:

Self-consistent determination of $n_0$ and $q$ for $R(\mathbf{x}-\mathbf{x}') = R\,\delta(\mathbf{x}-\mathbf{x}')$

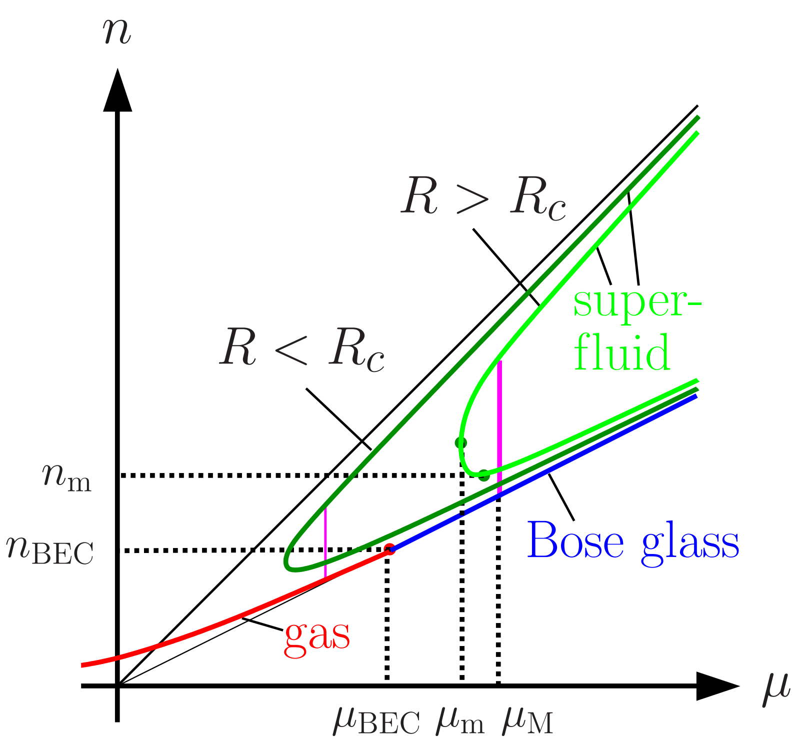
## Phase Classification:

| gas | Bose glass | superfluid |
|---|---|---|
| $q = n_0 = 0$ | $q > 0, n_0 = 0$ | $q > 0, n_0 > 0$ |

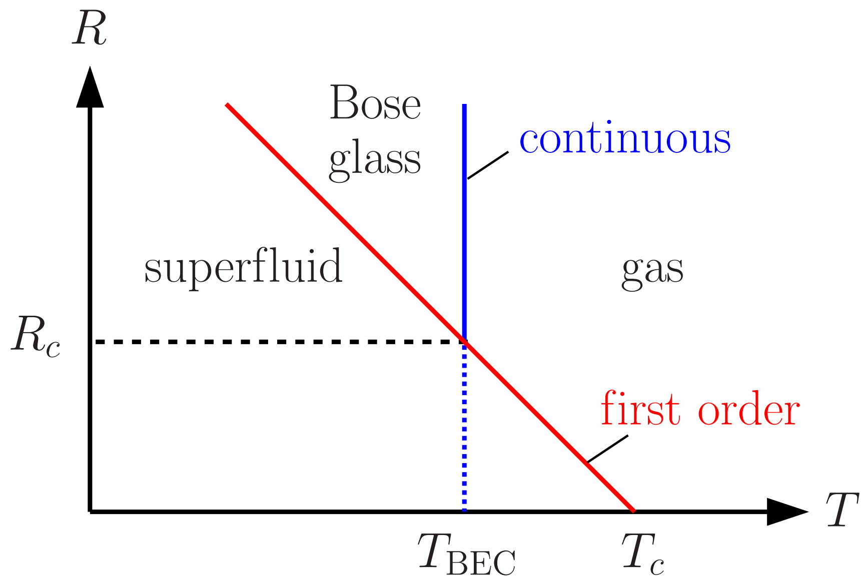
# 6.2 Hartree-Fock Results

**Isotherm:** $T = \text{const.}$

disorder strength $R$ = const.




**Phase Diagram:** $\mu = \text{const.}$




Graham and Pelster, Int. J. Bif. Chaos (in press)

# 7 Summary and Outlook

- **Frozen Disorder Potential**:

  arises both artificially (laser speckles) or naturally (wire trap)

- **Bosons:**

  local condensates in minima + global condensate + thermally excited

- **Localization Versus Transport:**

  disorder reduces superfluidity

- **Phase Diagram:**

  yet unknown for strong disorder

  Navez, Pelster, and Graham, Appl. Phys. B **86**, 395 (2007)

- **Disordered Bosons in Lattice:**

  Bose Glass versus Mott phase

  Krutitsky, Pelster, and Graham, New J. Phys. **8**, 187 (2006)

# 8 Acknowledgements



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| SFB/TR 12: Symmetries and Universality in Mesoscopic Systems | SPP 1116: Interactions in Ultra-Cold Atomic and Molecular Gases |

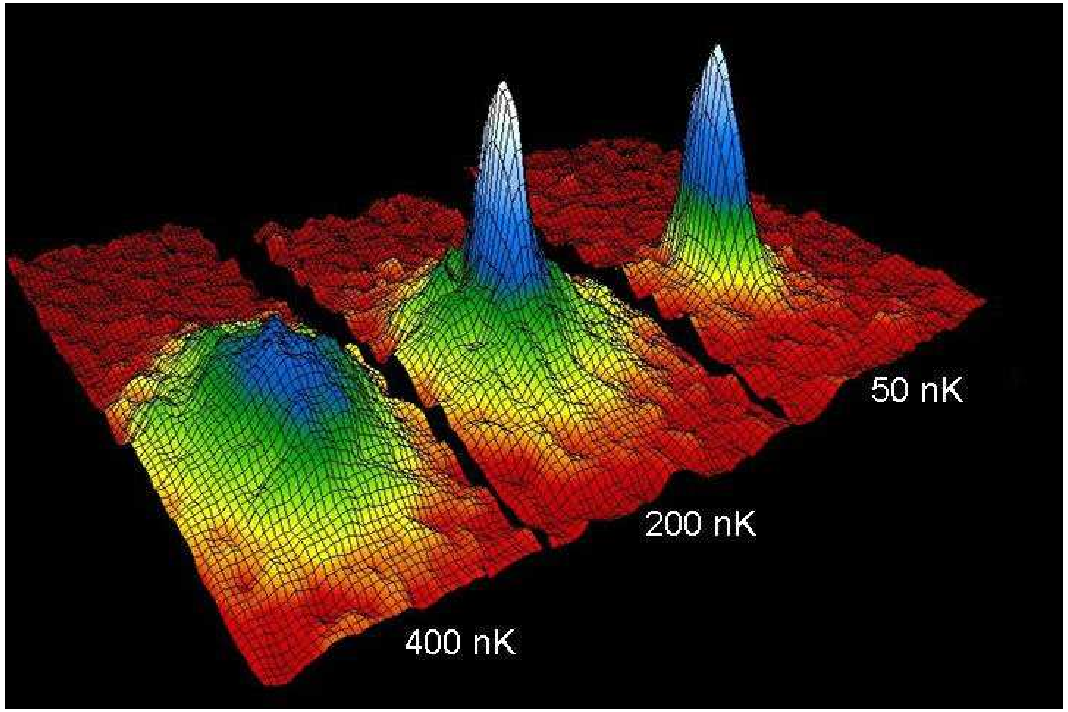
## 422nd Wilhelm and Else Heraeus Seminar

# Quo Vadis BEC?

**Bad Honnef, October 29 - 31, 2008**

*Scientific Coordinators:* Martin Holthaus (Oldenburg) and Axel Pelster (Duisburg-Essen)

*Research Topics:* BEC/BCS Crossover, Dipolar Gases, **Disorder**, Dynamics, Quantum Information, Spinor Bose and Fermi Gases, Strong Correlations, Tunneling




**http://www.theo-phys.uni-essen.de/tp/ags/pelster_dir/Heraeus/index.html**