Parametric and Geometric Resonances in Bose-Einstein Condensates

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Abstract

- *We study the phenomenon of parametric and geometric resonances in ultracold quantum gases by both analytical and numerical calculations.*
- *We examine a parametric model system for both an isotropic spherical and a cylindrical trap in order to study the resonance curves which have been observed in the experiment:*

  S. E. Pollack, D. Dries, R. G. Hulet K. M. F. Magalhães, E. A. L. Henn, E. R. F. Ramos, M. A. Caracanhas, and V. S. Bagnato: **"Collective excitation of a Bose-Einstein condensate by modulation of the atomic scattering length"**
  Phys. Rev. A 81, 053627 (2010)

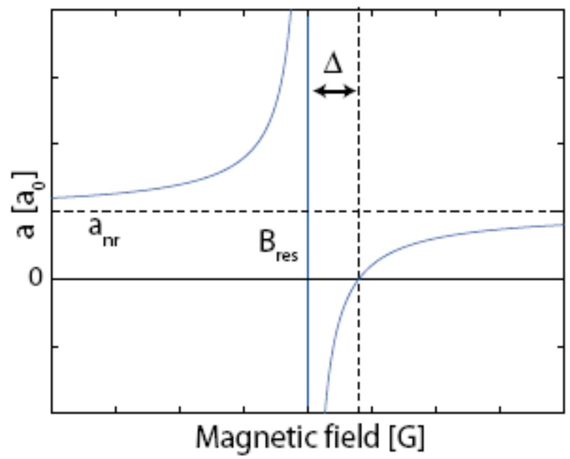
Parametric Resonance

Modulating s-wave scattering length via Feshbach resonance

- Experimental research groups V. S. Bagnato and R. G. Hulet Phys. Rev. A **81**, 053627 (2010).
- The modulation of the s-wave scattering length is obtained by introducing a small AC component to the bias field at the desired frequency and fixed amplitude:

$$B(t) = B_0 + b\ \cos\Omega\, t$$

$$a_s = a_{\rm nr}\left[1 - \frac{\Delta}{B(t) - B_{\rm res}}\right]$$



- If $b \ll | B_0 - B_{res} |$ then

$$a_s = a_{\rm av} + a\ \cos(\Omega\, t)$$

$$a_{\rm av} = a_{\rm nr}\left[1 - \frac{\Delta}{B_0 - B_{\rm res}}\right], a = \frac{a_{\rm nr} b\, \Delta}{(B_0 - B_{\rm res})^2}$$

Collective Oscillation

- In Trap BEC at *T = 0, GP equation*

$$\left[\frac{-\hbar^2}{2M}\nabla^2 + \mathrm{V_{Trap}}(\mathrm{r}) + \mathrm{g}|\psi(\vec{\mathrm{r}},\mathrm{t})|^2\right]\psi(\mathrm{r},\mathrm{t}) = \mathrm{i}\,\hbar\,\frac{\partial}{\partial \mathrm{t}}\psi(\mathrm{r},\mathrm{t})$$

- where Ψ(*r, t) is a condensate wave function.*
- An external axially symmetric harmonic trap potential

$$V(\vec{r}) = \tfrac{1}{2}m\omega_\rho^2\left(\rho^2 + \lambda^2 z^2\right)$$

- The contact interaction between atoms $g = 4\pi Na/m$
- The time dependent GP Equation action is

$$\mathcal{A}[\psi^*(\vec{x},t),\psi(\vec{x},t)]$$
$$= \int dt \int d^3x\, \mathcal{L}\left(\psi^*(\vec{x},t),\nabla\psi^*(\vec{x},t),\frac{\partial\psi^*(\vec{x},t)}{\partial t};\psi(\vec{x},t),\nabla\psi(\vec{x},t),\frac{\partial\psi(\vec{x},t)}{\partial t}\right)$$

- With the Lagrange density

$$\mathcal{L} = i\hbar\psi^*(\vec{x},t)\frac{\partial\psi(\vec{x},t)}{\partial t} - \frac{\hbar^2}{2M}\nabla\psi^*(\vec{x},t)\nabla\psi(\vec{x},t)$$
$$- V(\vec{x})\psi^*(\vec{x},t)\psi(\vec{x},t) - \frac{g(t)}{2}\psi(\vec{x},t)^2\psi^*(\vec{x},t)^2$$

- Gaussian ansatz:

$$\Psi(r,z,t) = \frac{1}{\sqrt{\pi^{3/2}u_r^2u_z}}\exp\left\{\frac{-1}{2u_r(t)^2} + ir^2\phi_r(t) + \frac{-1}{2u_z(t)^2} + ir^2\phi_z(t)\right\}$$

- The Lagrange function is:

$$L = \int d^3x\, \mathcal{L}$$

- Thus, the action defined

$$A[u_r, u_z] = \int dt L(u_r, \dot{u}_r, u_z, \dot{u}_z)$$

- This Leads to the Euler-lagrange equations

$$\frac{\partial L}{\partial u_r(t)} - \frac{d}{dt}\frac{\partial L}{\partial r(t)} = 0 \qquad \frac{\partial L}{\partial u_z(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}_z(t)} = 0$$

- The equations of motion when *p(t) = p + q cos ωt*

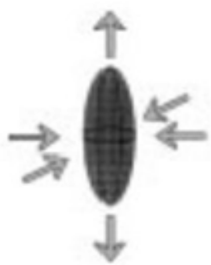
$$\ddot{u}_r + u_r = \frac{1}{u_r^3} + \frac{P(\tau)}{u_r^3 u_z} \qquad \ddot{u}_z + \lambda^2 u_z = \frac{1}{u_z^3} + \frac{P(\tau)}{2u_r^2 u_z^2}$$

- The time-independent solution *ur(t) = ur0, and uz(t) = uz0 for driving q = 0 is determined* from

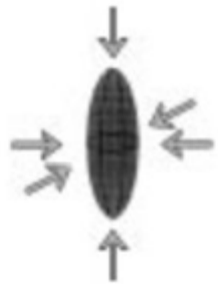
$$u_{r0} = \frac{1}{u_{r0}^3} + \frac{p}{u_{r0}^3 u_{z0}}, \qquad \lambda^2 u_{z0} = \frac{1}{u_{z0}^3} + \frac{p}{u_{r0}^2 u_{z0}^2}.$$

- The collective mode frequencies of the non-driven oscillations is given by

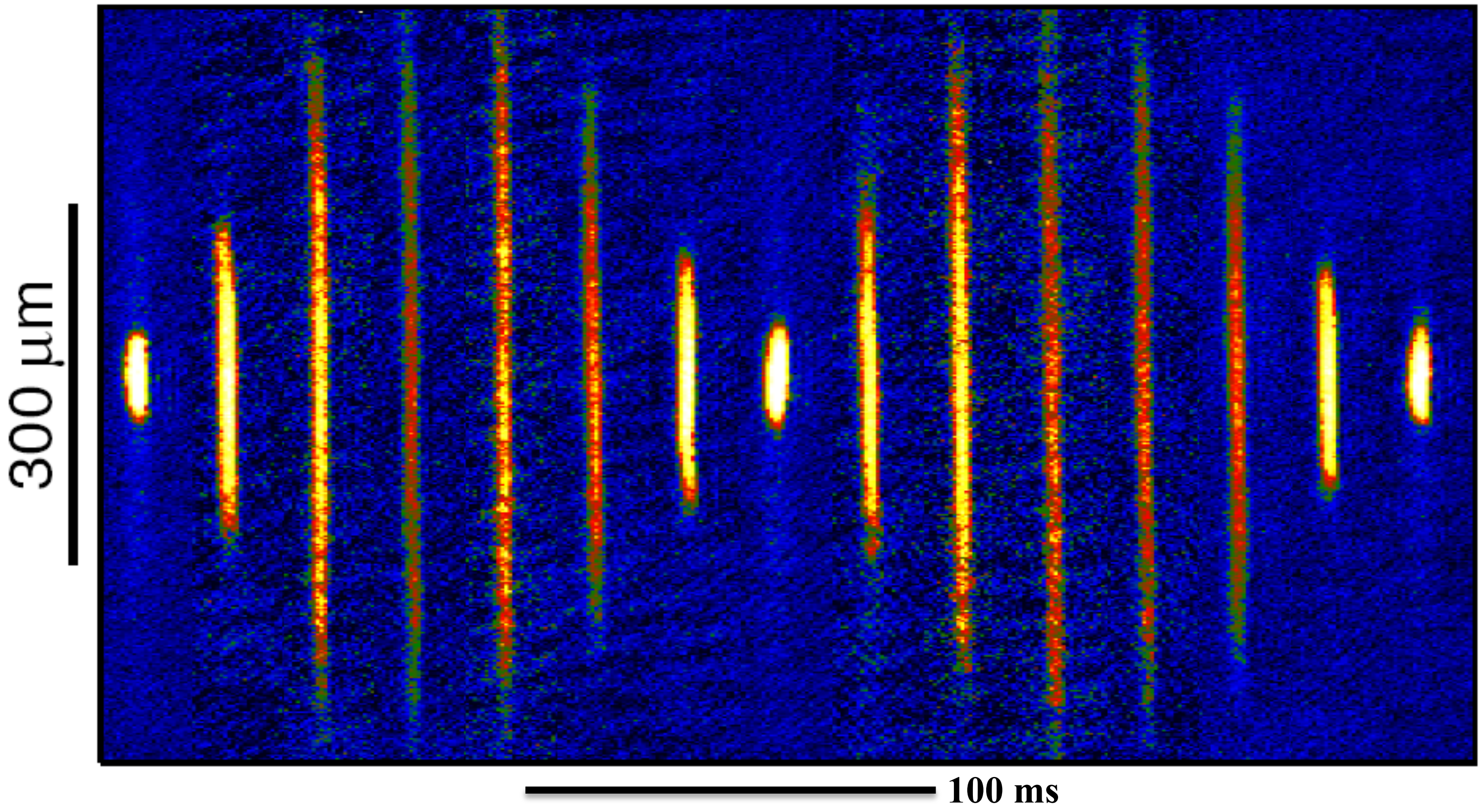
$$\omega_B, \omega_Q = \sqrt{2}\left[\left(1 + \lambda^2 - \frac{p}{4u_{r0}^2 u_{z0}^3}\right) \pm \sqrt{\left(1 - \lambda^2 + \frac{p}{4u_{r0}^2 u_{z0}^3}\right)^2 + 8\left(\frac{p}{4u_{r0}^3 u_{z0}^2}\right)^2}\,\right]^{\frac{1}{2}}.$$



Quadrupole mode

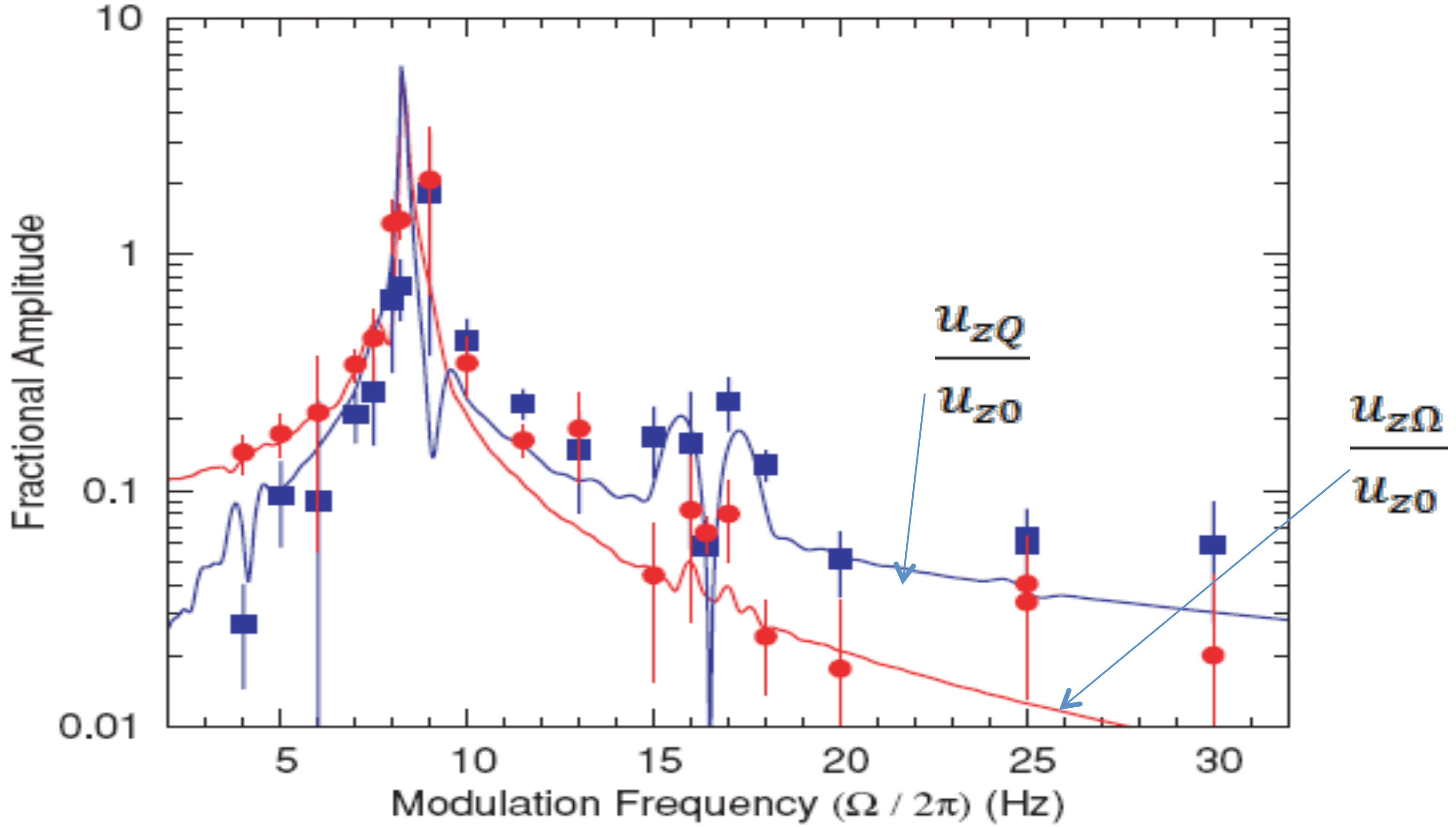


Breathing mode



Quadrupole oscillation excited by the modulation of the s-wave scattering length.

V. S. Bagnato and R. G. Hulet et al, Phys. Rev. A **81**, 053627 (2010).



The resonance curve for a parametric excitation of a quadrupole mode shows the excitation amplitude as a function of the external driving

Parametric Resonance in Spherical BEC

- At first, we study a parametric model system with one degree of freedom for an isotropic spherical trap, where the equation of motion for the condensate *u(t) width reads,*

$$\frac{d^2u(t)}{dt^2} + u(t) - \frac{1}{u(t)^3} - \frac{p}{u(t)^4} - \frac{q\cos(\Omega\, t)}{u(t)^4} = 0$$

- Its time-independent solution *u(t) = u0 for vanishing driving q = 0 is determined* from

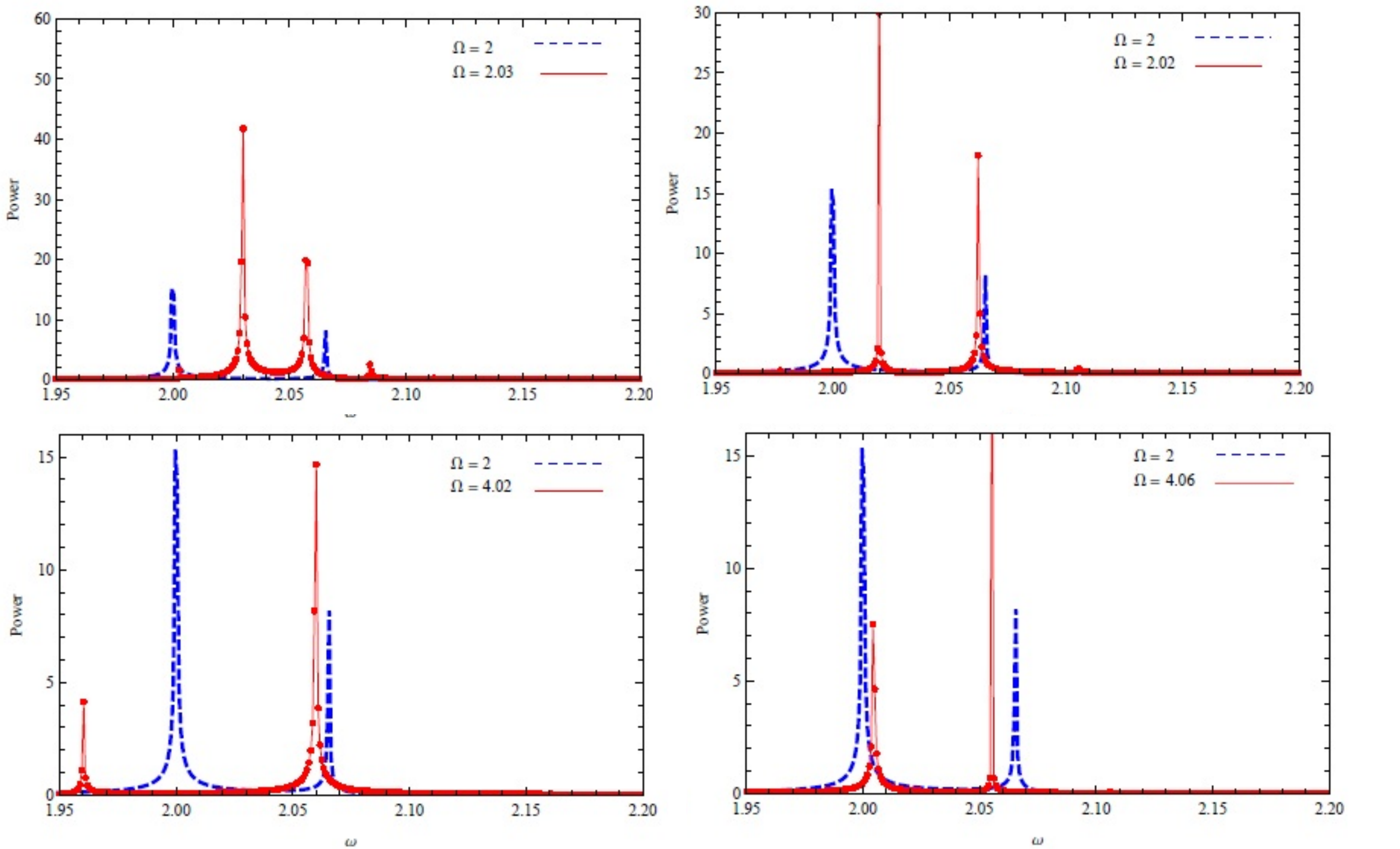
$$u_0 = \frac{1}{{u_0}^3} + \frac{P_0}{{u_0}^4}$$

- The collective mode frequency *ω0 of the non-driven oscillations is determined* by

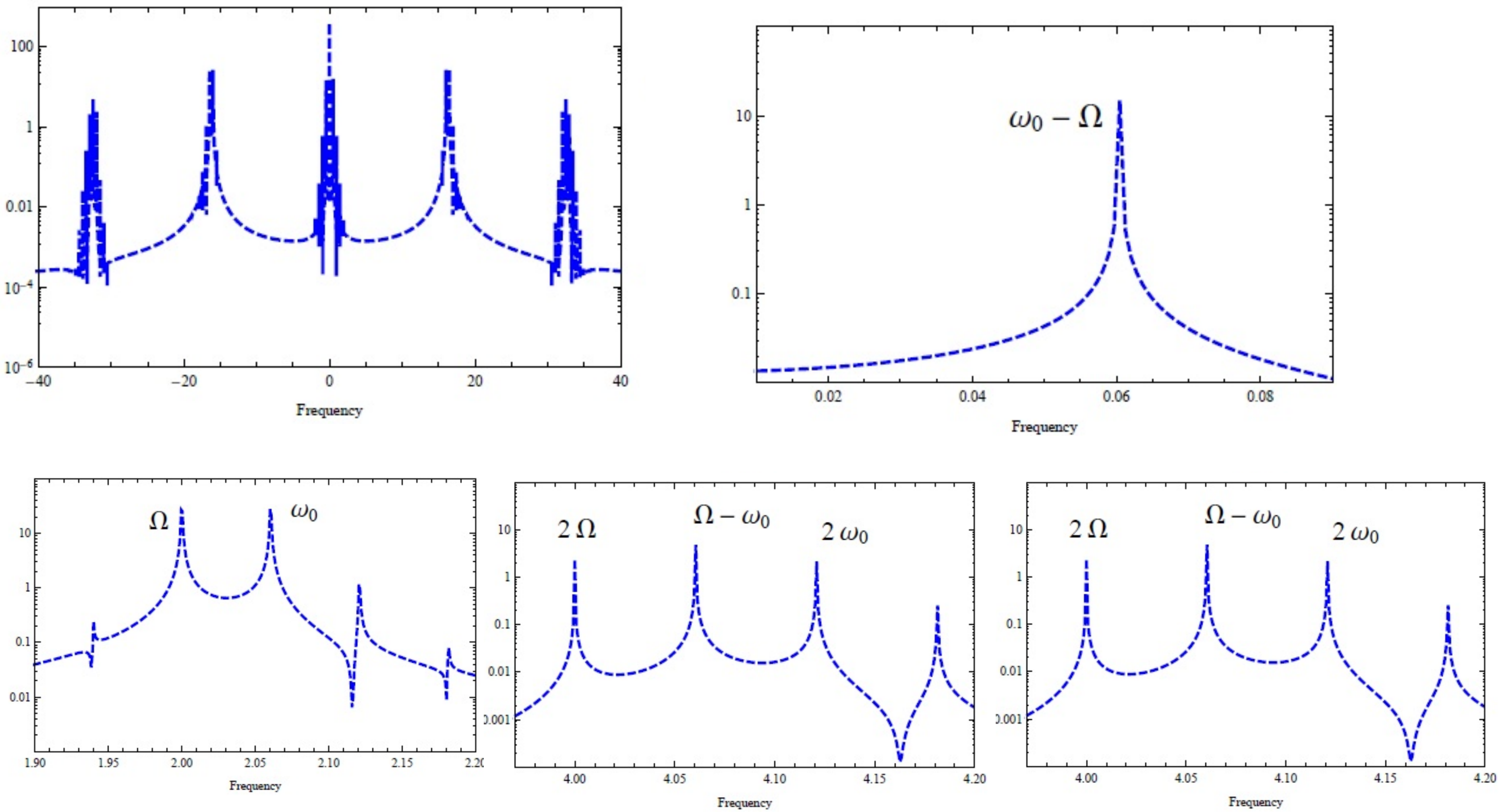
$${\omega_0}^2 = 1 + \frac{1}{{u_0}^4} + \frac{p}{{u_0}^5}$$

- *We solve the equation with initial condition*

$$u(0) = u_0, \dot{u}(0) = 0$$



Fourier spectrum of the condensate width in the vicinity of ω_0
for different values of Ω close to ω_0
and $2\omega_0$: *p = 0.4, q = 0.06.*



High-resolution Fourier spectrum of the condensate width:
p = 0.4, *q* = 0.1. *First plot gives an overview of the spectrum, while other plots zoom in to interesting* regions.

- In order to develop an analytical calculation that explains the nonlinear effect, we apply the standard Poincar´e-Lindstedt method. To this end we approximate by introducing a dimensionaless time scale *τ = ωt, so we get*

$$\omega^2 \frac{d^2 u(t)}{dt^2} + \omega_0{}^2 u(t) - \frac{1}{u(t)^3} - \frac{p}{u(t)^4} - \frac{q\cos(\Omega\, t)}{u(t)^4} = 0$$

- and we assume the following perturbative expansions in the modulation amplitude *q*, $u(\tau) = u_0 + q\, u_1(\tau) + q^2\, u_2(\tau) + \ q^3\, u_3(\tau)$

$$\omega = \omega_0 + q\, \omega_1(\tau) + q^2\, \omega_2(\tau) + \ q^3\, \omega_3(\tau)$$

- *We get system of linear differential equations*

$$\omega_0^2 \frac{d^2 u_1(\tau)}{dt^2} + \omega_0^2 u_1(\tau) = \frac{1}{u_0{}^4} \cos\Omega\,\tau$$

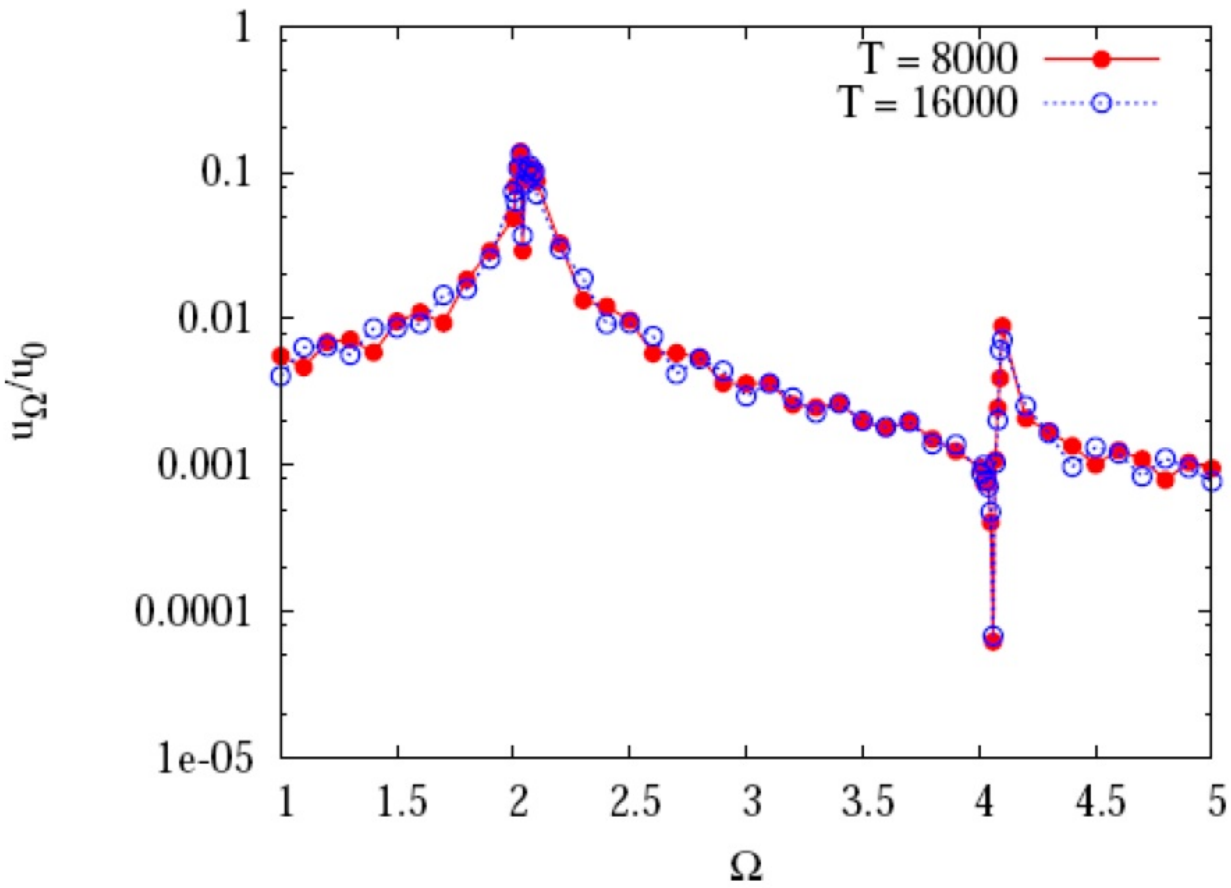
$$\omega_0^2 \frac{d^2 u_2(\tau)}{dt^2} + \omega_0^2 u_2(\tau)$$

$$= -2\omega_1\omega_0\, \ddot{u}_1(\tau) + \left(\frac{6}{u_0{}^5} + \frac{10P_0}{u_0{}^6}\right) u_1(\tau)^2 - \frac{4u_1(t)}{u_0{}^5}\cos\Omega\,\tau$$

$$\omega_0^2 \frac{d^2 u_3(\tau)}{dt^2} + \omega_0^2 u_3(\tau)$$

$$= \left(\frac{12}{u_0{}^5} + \frac{20P_0}{u_0{}^6}\right) u_1(\tau) u_2(\tau) - \left(\frac{10}{u_0{}^6} + \frac{20P_0}{u_0{}^7}\right) u_1{}^3(\tau)$$

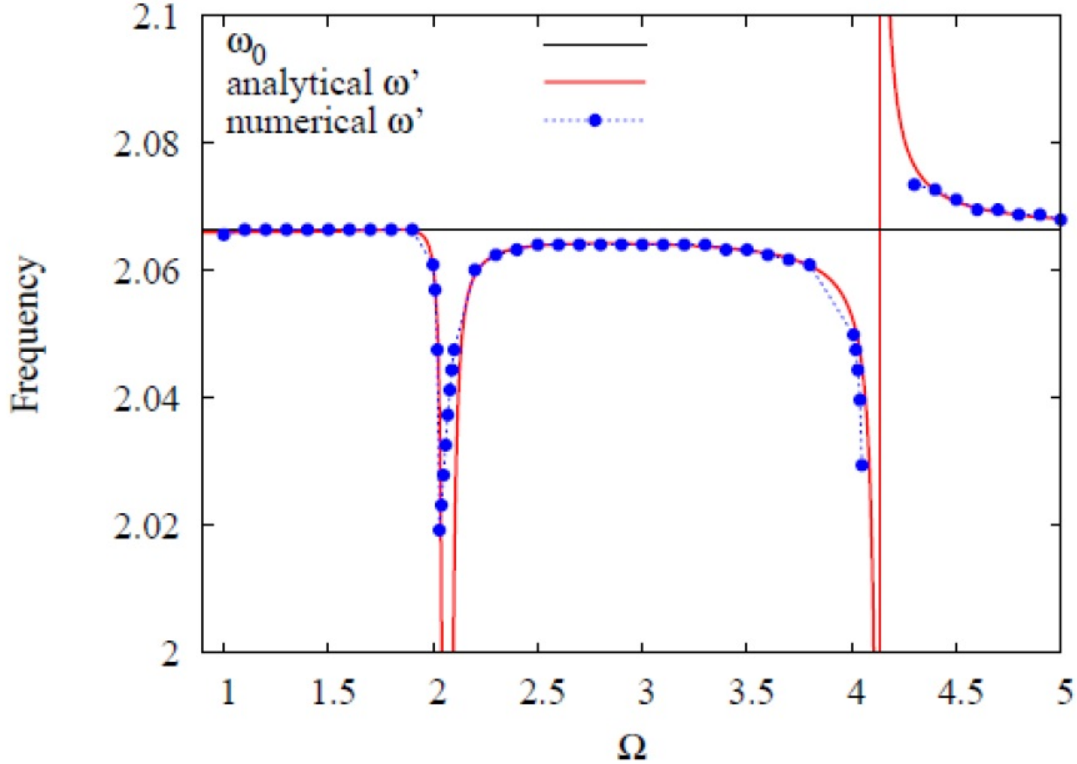
$$- \frac{4u_2(\tau)}{u_0{}^5}\cos\Omega\,\tau + \frac{10\, u_1{}^2(\tau)}{u_0{}^6}\cos\Omega\,\tau$$



Amplitude ration for the drive mode obtained using high resolution spectrum: *p* = 0.4, *q* = 0.06.

Let us consider the secular term, and absorb it into the frequency shift *$\omega = \omega_0 + \Delta\omega$ of the eigenmode in first order*

$$\omega = \omega_0 + q^2 \frac{f(\Omega)}{(\Omega^2 - \omega_0^2)(\Omega^2 - 4\omega_0^2)}$$



Frequency shift of the breathing mode with the driving frequency Ω. The blue line represents the analytical calculation up to third order of *q, while the red dots represent the numerics. We use p =* 0.4*, q* = 0.1.

Cylindrically Symmetric BEC

- Now, we turn to the case of axially symmetric BEC where we have to solve

$$\ddot{u}_r + u_r = \frac{1}{u_r^3} + \frac{p + q\cos\Omega\, t}{u_r^3 u_z}\,, \qquad \ddot{u}_z + \lambda^2 u_z = \frac{1}{u_z^3} + \frac{p + q\cos\Omega\, t}{u_r^2 u_z^2}$$

$$u_r(0) = 0, \dot{u}_r(0) = 0 \qquad u_z(0) = 0, \dot{u}_z(0) = 0$$

- time-independent solution *$u_r(t) = u_{r0}$, and $u_z(t) = u_{z0}$ for driving q = 0 is* determined from

$$u_{r0} = \frac{1}{u_{r0}^3} + \frac{p}{u_{r0}^3 u_{z0}} \qquad \lambda^2 u_{z0} = \frac{1}{u_{z0}^3} + \frac{p}{u_{z0}^2 u_{r0}^2}$$

$$\omega_B, \omega_Q = \sqrt{2}\left[\left(1 + \lambda^2 - \frac{p}{4u_{r0}^2 u_{z0}^3}\right) \pm \sqrt{\left(1 - \lambda^2 + \frac{p}{4u_{r0}^2 u_{z0}^3}\right)^2 + 8\left(\frac{p}{4u_{r0}^3 u_{z0}^2}\right)^2}\,\right]^{\frac{1}{2}}.$$

In order to study the frequencies of the collective modes analytically, we perform perturbative expansions in the modulation amplitude *q*

$$u_r(t) = u_{r0} + q_1 u_{r1}(t) + {q_1}^2 u_{r2}(t) + q^3 u_{r3}(t) + \cdots$$
$$u_z(t) = u_{z0} + q_1 u_{z1}(t) + {q_1}^2 u_{z2}(t) + q^3 u_{z3}(t) + \cdots$$

- we obtain a system of linear differential equations in the form

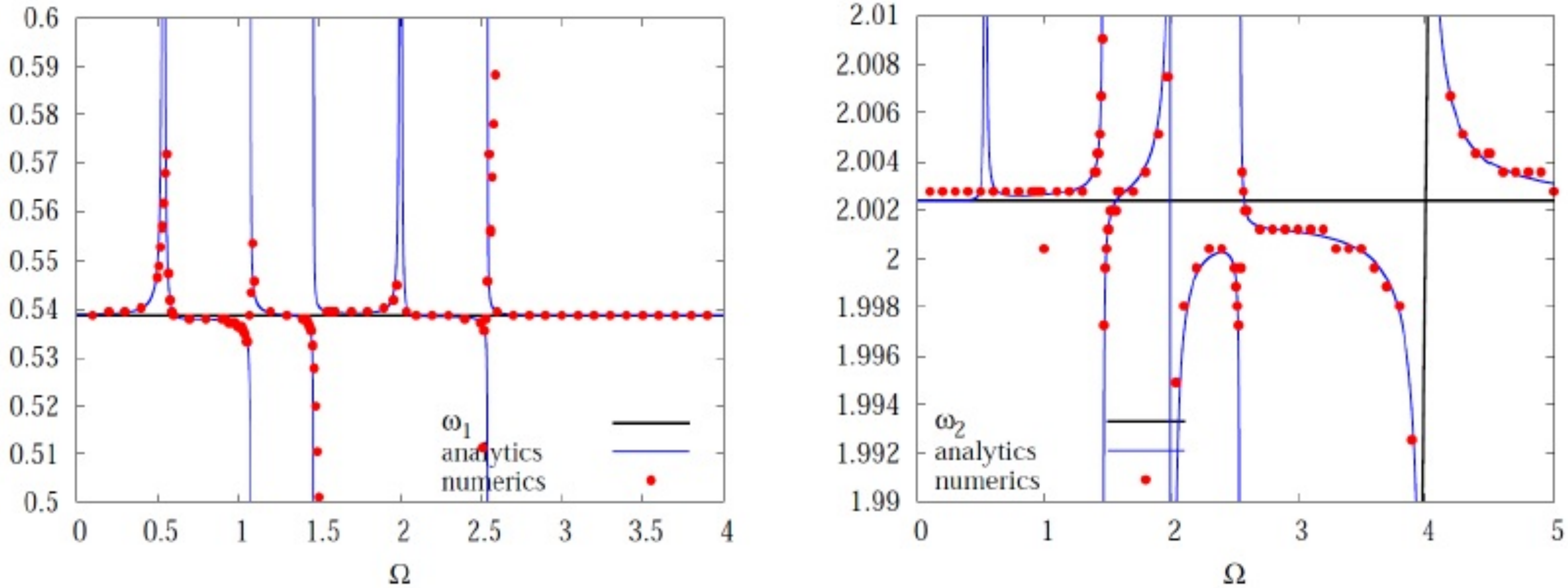
$$\ddot{u}_{rn} + m_1 u_{rn} + m_2 u_{zn} + f_{rn} = 0$$
$$\ddot{u}_{zn} + 2m_2 u_{rn} + m_3 u_{zn} + f_{zn} = 0$$

- We decouple the equations using the linear transformation

$$u_{rn}(t) = x_n(t) + y_n\,(t)$$
$$u_{zn}(t) = c_1 x_n(t) + c_2 y_n\,(t)$$

The secular terms appear at the level $n = 3$. *Let us now consider the secular* terms, and absorb them into a shift $\omega = \omega_Q + \Delta\omega_Q$ *of the eigenmode frequency* in first order, so we get the frequency shift from the expression

$$\Delta\omega_Q = -\frac{C_Q q^2}{2\ \omega_Q}$$



Frequency shift of the quadrupole mode frequency on the left side and breathing mode frequency on the right, respectively, versus driving frequency for $p = 1$, $q = 0.2$, $\lambda = 0.3$.

Geometric Resonance

- To study a geometric resonance in a BEC for a cylindrical trap, we start from

$$\ddot{u}_r(t) + u_r(t) = \frac{1}{u_r^3(t)} + \frac{p}{u_r^3(t) u_z(t)}$$

$$\ddot{u}_z(t) + \lambda^2 u_z(t) = \frac{1}{u_z^3(t)} + \frac{p}{u_r^2(t) u_z^2(t)}$$

- In this case we do not have a parametric excitation. We try to excite the quadrupole mode only by solving with the initial conditions

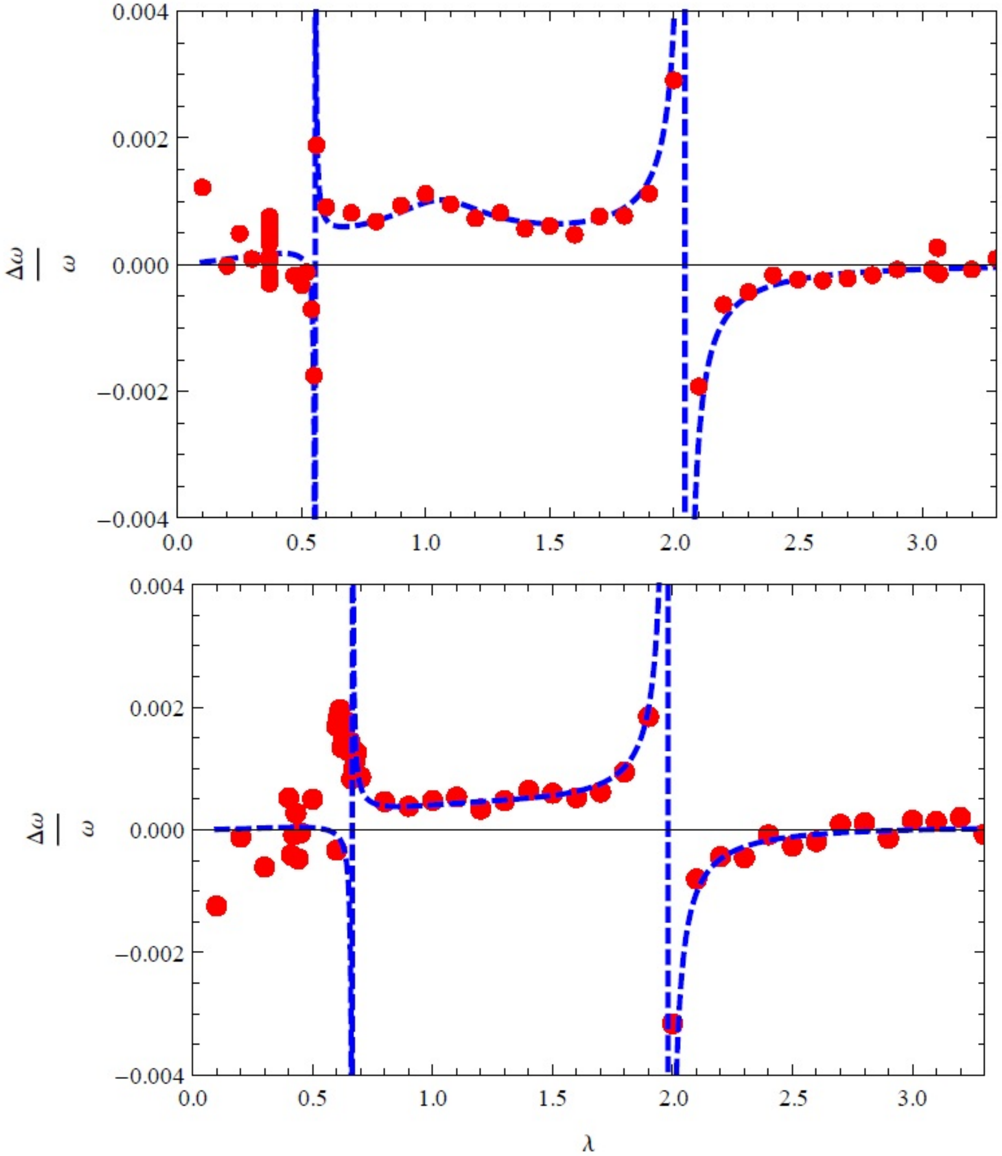
$$\boldsymbol{u}(\mathbf{0}) = \boldsymbol{u}_0 + \varepsilon\, \boldsymbol{u}_Q$$

$$\dot{\boldsymbol{u}}(\mathbf{0}) = \mathbf{0}$$

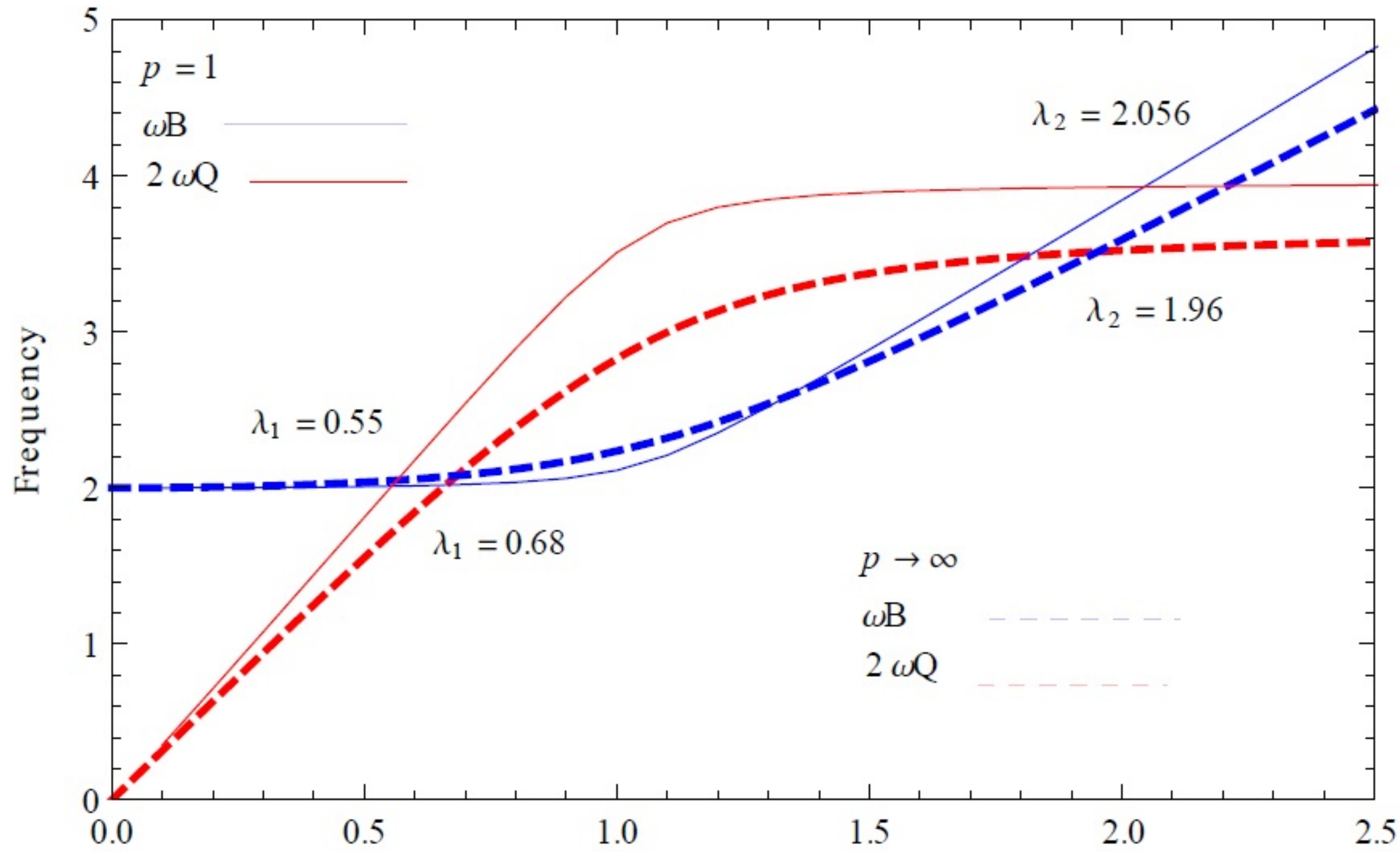
- where $\mathbf{u}_Q$ is the quadrupole mode eigenvector
- We apply the standard Poincar´e-Lindstedt method to perform perturbative expansions for small ε

- We calculate the frequency shift to be:

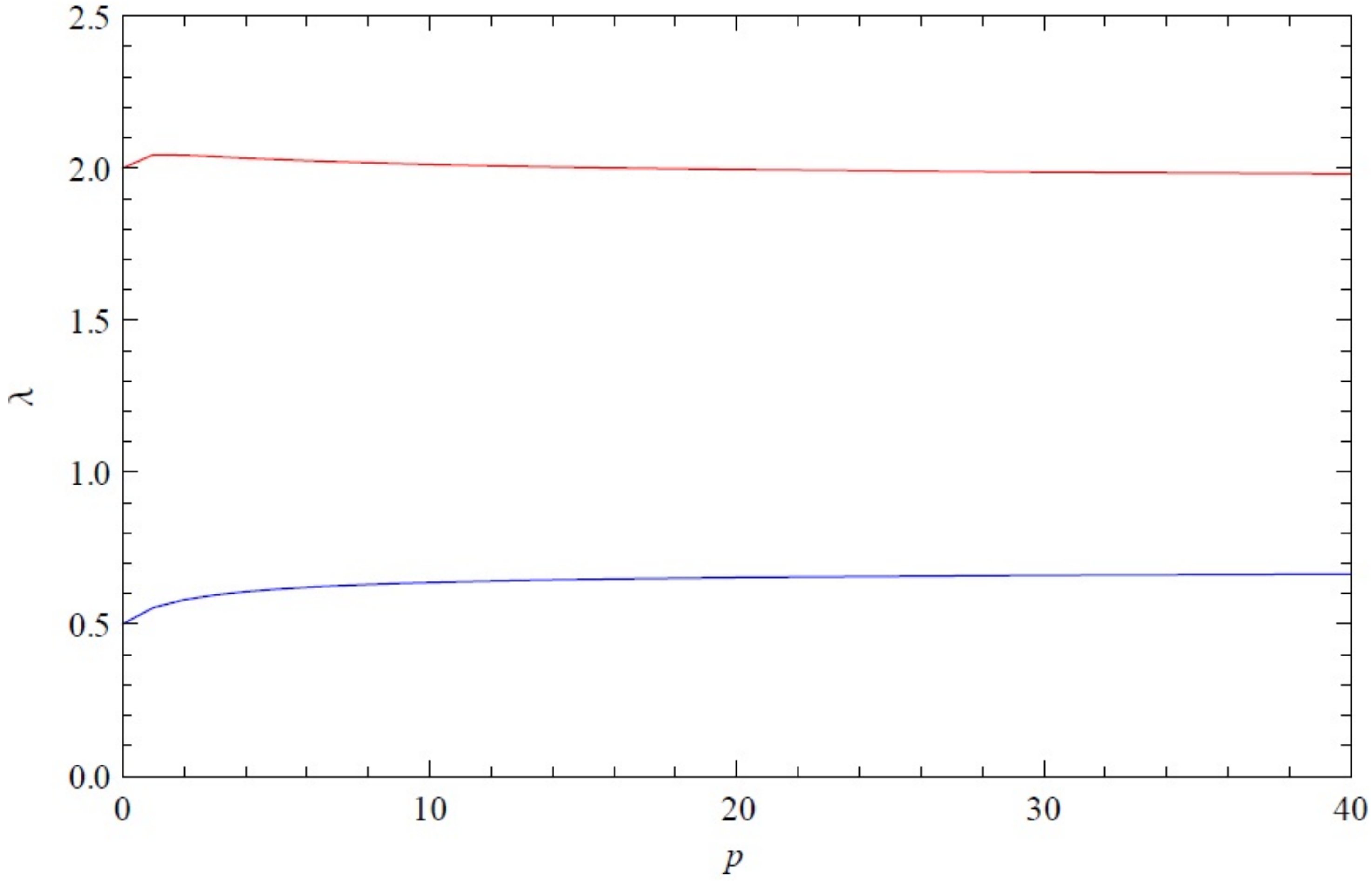
$$\frac{\Delta\omega_Q}{\omega_Q} = \varepsilon^2 \frac{f(\omega_Q, \omega_B, u_{r0}, \omega_{z0}, p, \lambda)}{2\omega_Q^2 ({\omega_B}^2 - 4\omega_Q^2)^2}$$



Frequency shift of quadrupole mode versus trap aspect ratio λ. Upper plot for $p = 1$ and lower plot for $p \to \infty$ and $\varepsilon = 0.1$. The red dots are numerically obtained values, while the blue dots represent analytical results calculated in the third order of perturbation theory.

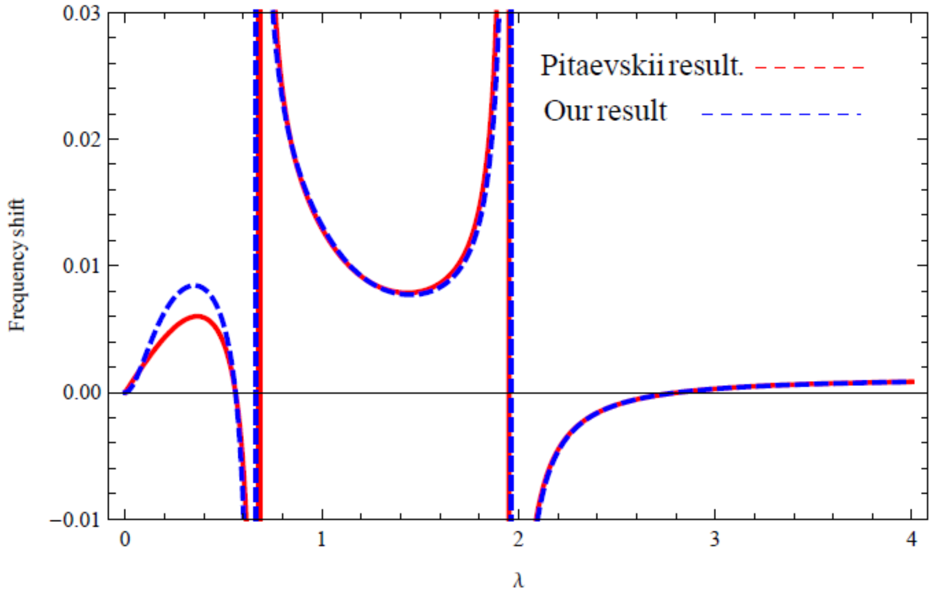


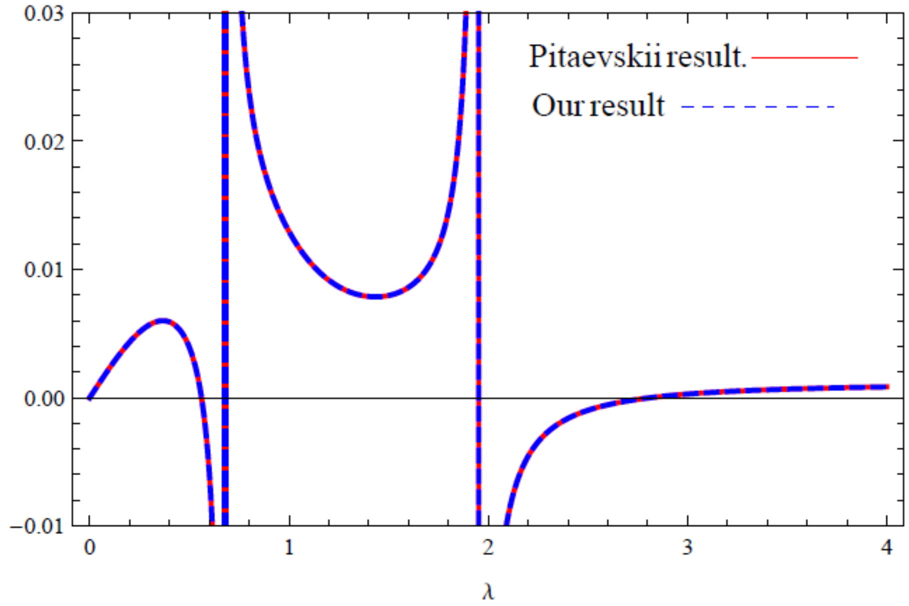
Frequencies of breathing and quadrupole mode versus trap aspect ratio λ.



p − λ plane

- F. Dalfovo, C. Minniti and L.P. Pitaevskii1, Phys. Rev. A **56, 4855 (1997)**





Comparison of our results with Pitaevskii's results.

Outlook

- Geometric resonances in a system with anharmonic trap, where the anharmonicity can play a role of the parameter.
- Parametric resonances due to modulation of a trapping frequency, or anharmonicity, or interaction strength.
- Geometric and parametric resonances for dipolar systems.
- Geometric resonances in systems with 2-body + 3-body interactions.

Thank you For your

Attention